The Penrose Transform for Einstein-Weyl and Related Spaces

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PhD
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1996

Abstract

A holomorphic Penrose transform is described for Hitchin's correspondence between complex Einstein-Weyl spaces and "minitwistor" spaces, leading to isomorphisms between the sheaf cohomologies of holomorphic line bundles on a minitwistor space and the solution spaces of some conformally invariant field equations on the corresponding Einstein-Weyl space. The Penrose transforms for complex Euclidean 3-space and complex hyperbolic 3-space, two examples which have preferred Riemannian metrics, are explicitly discussed before the treatment of the general case.

The non-holomorphic Penrose transform of Bailey, Eastwood and Singer, which translates holomorphic data on a complex manifold to data on a *smooth* manifold, using the notion of involutive cohomology, is reviewed and applied to the non-holomorphic twistor correspondences of four homogeneous spaces: Euclidean 3-space, hyperbolic 3-space, Euclidean 5-space (considered as the space of trace-free symmetric 3×3 matrices) and the space of non-degenerate real conics in complex projective plane. The complexified holomorphic twistor correspondences of the last two cases turn out to be examples of a more general correspondence between complex surfaces with rational curves of self-intersection number 4 and their moduli spaces.

Declaration

I hereby declare that the thesis is composed by me and is my own work.

Acknowledgments

I would like to thank my supervisor Dr. T. N. Bailey for his guidance, encouragement and comments throughout the period of my study at Edinburgh University. Thanks are also due to the Department of Mathematics and Statistics for providing such a nice environment in which to work, and to the ORS Awards Scheme for partial financial support.

Table of Contents

Introduction **1**

0. Background Material **6**

0.1 Basic material on manifolds and sheaves 6

0.2 Twistor correspondences 9

0.3 Homogeneous vector bundles 11

0.4 Conformal structure and projective structure 12

0.5 Spinor index notation 13

0.6 Some results on direct images 14

1. The Holomorphic Penrose Transform **17**

2. A Holomorphic Penrose Transform for Euclidean 3-space **20**

2.1 Correspondence and set-up 20

2.2 Results 27

2.3 The $\lambda = 0$ case 30

Appendix: Explicit contour integral formula for $\nabla^2\phi = \lambda^2\phi$ 31

3. A Holomorphic Penrose Transform for Hyperbolic 3-Space **33**

3.1 Correspondence and set-up 33

3.2 Results 38

4. The Penrose Transform for Einstein-Weyl Spaces **42**

4.1 Geometry of EW spaces 42

4.2 Hitchin's correspondence 46

4.3 Spinorial treatment of EW spaces 47

4.4 The Penrose transform 53

4.5 Mini-ambitwistor spaces 62

4.6 Relation between $\mathcal{N}$ and $\mathcal{T}$ 65

5. The Non-holomorphic Penrose Transform **68**

5.1 Involutive structure 69

5.2 Non-holomorphic Penrose transform 73

5.3 Involutive structure on homogeneous spaces 76

5.4 Non-holomorphic Penrose transform in the homogeneous case . . . 80

Appendix 1: Proofs of some propositions 84

Appendix 2: Realization of holomorphic Penrose transforms 87

6. A Non-holomorphic Penrose Transform for Euclidean 3-Space **88**

6.1 Correspondence and set-up 88

6.2 Results 91

7. A Non-holomorphic Penrose Transform for Hyperbolic 3-Space **96**

7.1 Correspondence and set-up 96

7.2 Results 100

8. A Penrose Transform for $\mathbb{R}^5$ **106**

8.1 Holomorphic twistor correspondence 106

8.2 Non-holomorphic twistor correspondence 109

8.3 The Transform 114

Appendix: An alternative approach 120

9. A Penrose Transform for M^5 **122**

9.1 Holomorphic twistor correspondence 122

9.2 Non-holomorphic twistor correspondence 124

9.3 The Transform 128

Appendix: Some formulae 133

10. A Generalized Correspondence **137**

10.1 Complex surfaces with curves of normal bundle $\mathcal{O}(4)$ and their moduli spaces 137

10.2 A Generalization 140

Bibliography **142**

Introduction

History

In 1966, Penrose [27] discovered that solutions of massless field equations on Minkowski space could be expressed as contour integrals of free holomorphic functions over lines in 3 dimensional complex projective space. This idea of using contour integral formulae to obtain solutions of field equations can in fact be traced back to Whittaker [36] in 1902 (cf. also Bateman [6]). The freedom in the function for a fixed solution was later realized to be exactly the freedom of a Čech representative of a sheaf cohomology class, and a formal mechanism was set up by Eastwood, Penrose and Wells to prove isomorphisms between sheaf cohomology groups over a region of the projective space and solutions of massless field equations over a region of spacetime [8].

The *Penrose transform* was originally ascribed to this particular mechanism for 4 dimensional flat spacetime. However, it has undergone a considerable amount of generalization and refinement since [8]. Two most important generalized Penrose transforms are:

Holomorphic Penrose transform: This is a mechanism which relates the Dolbeault cohomology group $H^r(Z, \mathcal{O}(V))$ of a complex manifold Z, where $V \to Z$ is a holomorphic vector bundle, to solutions of systems of linear partial differential equations on another complex manifold X, where X is a moduli space of compact complex submanifolds of Z, more precisely, we require that X and Z are related by a *holomorphic twistor correspondence* (see Section 0.2 and Chapter 1 for further conditions).

There are two main areas in the applications of this transform: **1.** The generalization of [8] to general *homogeneous* correspondences, i.e. correspondences for which spaces involved are all homogeneous spaces of a symmetry group. This is often referred to as the *flat* theory. In [4], the Penrose transform over generalized flag varieties is treated in great detail. Other examples include Penrose transforms for complex Euclidean 3-space and complex hyperbolic 3-space, cf. [18] and Chapters 2 and 3. **2.** The construction of the Penrose transform for the 'curved' version of the flat theory, examples of which include [1,2,9] and Chapter 4.

Non-holomorphic Penrose transform: This is a mechanism which, given that a complex manifold Z and a *smooth* manifold X are related by a correspondence with suitable conditions satisfied (see Chapter 5), relates $H^r(Z, \mathcal{O}(V))$, where V is again a holomorphic vector bundle over Z, to solutions of linear PDE's on X.

Examples of this transform include Hitchin [14], LeBrun [24], Salamon [31]. There are also works of representation theorists which are of this nature, see [3,10] for brief descriptions and references. More recently, Bailey, Eastwood and Singer [3] developed a mechanism which unified all the examples cited above, and could be rightly called *the non-holomorphic Penrose transform.* In addition to examples considered in [3], Chapters 6–9 are also devoted to applications of this mechanism.

It seems natural, following [3], to discuss this method using the idea of involutive structures [34], and since no treatment of this has appeared yet in the literature, we give a comprehensive discussion in Chapter 5.

Outline of thesis

- **Reviews:** Chapters 0,1,5.

In Chapter 0, we briefly review some basic material (sheaves, twistor correspondences, homogeneous vector bundles, conformal and projective geometries, spinor index notation, and some results on direct images) which shall be assumed in latter chapters.

In Chapter 1 we introduce the holomorphic Penrose transform, whose main structure was basically completed in [8].

In an earlier version of [3], a general Penrose transform is discussed, using the idea of involutive structures. Since this has not yet appeared in the literature, we provide our own treatment in Chapter 5, extending their results. For a homogeneous correspondence, the Penrose transform can in fact be computed in terms of Lie groups and Lie algebras. This is discussed in the latter part of Chapter 5. The relationship between a holomorphic Penrose transform and a non-holomorphic Penrose transform is discussed in Appendix 2 of this chapter.

• **Holomorphic Penrose transforms:** Chapters 2–4.

The applications of the holomorphic Penrose transform to cases where the target spaces X are complex Euclidean 3-space $\mathbb{C}^3$ and complex hyperbolic 3-space $\mathbb{H}$ first appeared in Jones thesis [18] and were treated as symmetry reductions of [8]. We give an alternative treatment using only the intrinsic geometry. Moreover, for the $\mathbb{C}^3$ case, the transform as described by Jones does not consider all the homogeneous line bundles on the twistor space. There is a complex parameter for which only the zero value is considered by Jones.

As the twistor correspondences of Chapter 2 and 3 are examples of Hitchin's correspondence between minitwistor spaces and Einstein-Weyl spaces [16], we investigate the holomorphic Penrose transform for general Einstein-Weyl spaces in Chapter 4. We also consider the holomorphic Penrose transform for the correspondence between a complex conformal 3-manifold and its mini-ambitwistor space — the space of null geodesics. The relation between a mini-ambitwistor space and a minitwistor space is discussed at the end of this chapter.

• **Non-holomorphic Penrose transforms:** Chapters 6–9

A non-holomorphic twistor correspondence for Euclidean 3-space $\mathbb{R}^3$ has been used by Whittaker [36] and Hitchin [15] to solve the Laplace equation and the Bogomolny equation respectively. However, the formal application of the mechanism of Chapter 5 to obtain full results has been lacking. In particular, as in the $\mathbb{C}^3$ case, the homogeneous bundles on the twistor space are classified by an integer and a complex value parameter. We discuss this transform in full generality in Chapter 6.

For hyperbolic 3-space H, a non-holomorphic Penrose transform was first given in [3], where H is considered as the homogeneous space $SO_0(3,1)/SO(3)$. However, in Chapter 7, by considering a different symmetry group $SL(2,\mathbb{C})$, we are able to obtain additional results over [3], as we have additional line bundles which relate to fields taking values in spin representations.

The Penrose transforms of Chapters 6 and 7 are in fact real forms of that of Chapters 2 and 3. Note that, however, for the case of H, we obtain more results than that of $\mathbb{H}$, as they have different twistor spaces. Combining the results of these chapters, we obtain that every smooth eigenfunction of the Laplacian on $\mathbb{R}^3$ (or H) extends to a holomorphic eigenfunction of the holomorphic Laplacian on $\mathbb{C}^3$ ($\widehat{\mathbb{H}}$ respectively), where $\widehat{\mathbb{H}}$ is an open subset of $\mathbb{H}$, see Section 7.2. The same applies to other equations involved.

In Chapter 8, we identify Euclidean 5-space $\mathbb{R}^5$ with the trace-free symmetric product of $\mathbb{R}^3$, then set up a correspondence and discuss its non-holomorphic Penrose transform. Neither this nor its corresponding holomorphic Penrose transform for the complexified space $\mathbb{C}^5$ are known to be in the literature before.

In Chapter 9, a non-holomorphic Penrose transform for the space of non-degenerate *real* conics in complex projective 2-plane is discussed. This is the first application of the non-holomorphic Penrose transform to a non-compact Riemannian symmetric space other then hyperbolic space.

- **A generalized correspondence:** Chapter 10

The holomorphic counterparts of the non-holomorphic twistor correspondences of Chapters 8 and 9 both have a twistor space admitting rational curves of normal bundle $\mathcal{O}(4)$. In Chapter 10, we discuss the general correspondence between complex surfaces admitting curves of normal bundle $\mathcal{O}(4)$ and their moduli spaces. This is then generalized to a correspondence where $\mathcal{O}(4)$ is replaced by $\mathcal{O}(2n)$, $n \geq 1$ (when $n = 1$, this becomes Hitchin's correspondence).

It is possible that the correspondences of this chapter are dimension reductions of the twistor correspondences for manifolds with a torsion-free quaternionic conformal structure [2]. Furthermore, there exist holomorphic Penrose transforms for such correspondences, but we have not computed details.

Chapter 0

Background Material

0.1 Basic material on manifolds and sheaves

Given a smooth manifold M and a smooth vector bundle V over M, a complex manifold N and a holomorphic vector bundle V' over N, we write, for $p, q \geq 0$,

$\mathcal{E}^p_M$:= sheaf of germs of smooth p-forms on M,

$\mathcal{E}^p_M(V)$:= sheaf of germs of smooth V-valued p-forms on M,

Ω^p_N := sheaf of germs of holomorphic p-forms on N,

$\Omega^p_N(V')$:= sheaf of germs of holomorphic V'-valued p-forms on N,

$\mathcal{E}^{p,q}_N$:= sheaf of germs of smooth forms of type (p, q) on N,

$\mathcal{E}^{p,q}_N(V')$:= sheaf of germs of sections of smooth V'-valued (p, q)-forms on N,

and $\mathcal{E}_M := \mathcal{E}^0_M$, $\mathcal{E}_M(V) := \mathcal{E}^0_M(V)$, $\mathcal{O}_N := \Omega^0_N$ and $\mathcal{O}_N(V') := \Omega^0_N(V')$. Note forms are taken to be complex valued and V, V' are complex vector bundles.

Pullback sheaves and relative de Rham sequence

Definition 0.1 *Let $f : Y \to B$ be a surjective map of maximal rank of complex manifolds, and $\mathcal{S}$ be a sheaf on B. The topological pullback sheaf $f^{-1}(\mathcal{S})$ on Y is the sheaf generated by the following presheaf*

$$U \to \mathcal{S}(f(U)), \quad U \textit{ is open in } Y.$$

Definition 0.2 *Let $\mathcal{S}$ be a sheaf of $\mathcal{O}_B$-module on B. The (analytic) pullback of $\mathcal{S}$ on Y is defined as*

$$f^*\mathcal{S} := f^{-1}(\mathcal{S}) \otimes_{f^{-1}\mathcal{O}_B} \mathcal{O}_Y.$$

Definition 0.3 *The sheaf of holomorphic relative p-forms Ω^p_f on Y is defined by the exactness of the following sequence*

$$0 \to f^*\Omega^1_B \wedge \Omega^{p-1}_Y \xrightarrow{i} \Omega^p_Y \xrightarrow{\pi} \Omega^p_f \to 0, \tag{0.1}$$

where i is the natural inclusion map and π is the quotient map.

Note there is a natural map

$$d_f : \Omega^p_f \to \Omega^{p+1}_f$$

which is given by $\pi \circ d$, where d is the usual exterior differential operator $d : \Omega^p_Y \to \Omega^{p+1}_Y$ and π is the quotient map $\Omega^{p+1}_Y \to \Omega^{p+1}_f$ in (0.1).

Lemma 0.4 *Let $k = dim(Y) - dim(B)$, then the relative holomorphic de Rham sequence*

$$0 \to f^{-1}\mathcal{O}_B \longrightarrow \mathcal{O}_Y \xrightarrow{d_f} \Omega^1_f \to \ldots \to \Omega^k_f \to 0 \tag{0.2}$$

is an exact sequence of sheaves on Y.

Let V be a holomorphic vector bundle on B. One can obtain a resolution of $f^{-1}\mathcal{O}_B(V)$ by tensoring (0.2) with $\otimes_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{O}_B(V)$, cf. [8], which yields

Lemma 0.5 *The (twisted) relative holomorphic de Rham sequence*

$$0 \to f^{-1}\mathcal{O}_B(V) \longrightarrow \mathcal{O}_Y(f^*V) \xrightarrow{d_f} \Omega^1_f(f^*V) \to \ldots \to \Omega^k_f(f^*V) \to 0 \tag{0.3}$$

is exact, where $\Omega^p_f(f^*V)$ denotes $\Omega^p_f \otimes f^*\mathcal{O}_B(V)$.

Spectral sequence of a differential sheaf

Definition 0.6 *A graded sheaf $\mathcal{S}^*$ is a sequence of sheaves $\{\mathcal{S}^n\}$, $n \in Z$. A differential sheaf is a graded sheaf with a homomorphism of sheaves $d : \mathcal{S}^n \longrightarrow \mathcal{S}^{n+1}$, which satisfies $d^2 = 0$.*

Definition 0.7 *A differential sheaf $\mathcal{S}^*$ on a manifold M is acyclic if $H^q(M, \mathcal{S}^p) = 0$, for $q \geq 1$, $p \geq 0$.*

Definition 0.8 *Let*

$$0 \longrightarrow \mathcal{S} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \ldots \longrightarrow A^n \longrightarrow \cdots \tag{0.4}$$

be a differential sheaf on M, then it is a <u>resolution</u> *of $\mathcal{S}$ if (0.4) is an exact sequence of sheaves.*

Theorem 0.9 *One has*

(i) *if (0.4) is a resolution of $\mathcal{S}$, then there is a spectral sequence*

$$E_1^{p,q} = H^q(M, A^p) \Longrightarrow H^r(M, \mathcal{S}),$$

(ii) *if (0.4) is acyclic, then there is a spectral sequence*

$$E_2^{p,q} = H^p(M, \mathrm{H}^q(A^*)) \Longrightarrow H^r(\Gamma(M, A^*)),$$

where $\mathrm{H}^q(A^*) := \dfrac{\ker(A^q \longrightarrow A^{q+1})}{\mathrm{im}(A^{q-1} \longrightarrow A^q)}$.

Proof: see e.g. [35]. □

Corollary 0.10 (Abstract de Rham theorem) *If (0.4) is both a resolution and acyclic, then*

$$H^p(M, \mathcal{S})) \cong H^p(\Gamma(M, A^*)).$$

Leray spectral sequence

Definition 0.11 *If $f : Y \to B$ is a continuous mapping of topological spaces and $\mathcal{S}$ is a sheaf on Y, then the direct image sheaf $f_*^q\mathcal{S}$ on B is the sheaf generated by the following presheaf:*

$$U \to H^q(f^{-1}(U), \mathcal{S}), \quad U \text{ is open in } B.$$

Theorem 0.12 (Leray spectral sequence) *If $f : Y \to B$ is a proper surjective mapping of topological spaces, $\mathcal{S}$ is a sheaf on Y. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(B, f_*^q\mathcal{S}) \Longrightarrow H^r(Y, \mathcal{S}).$$

For a proof, see, for instance, [35].

Corollary 0.13 *If S is coherent (thus $f^q_* S$ is coherent by the Direct Image Theorem, cf. [12]) and B is Stein (i.e. for any coherent sheaf $\mathcal{F}$ on B, $H^p(B, \mathcal{F}) = 0$ for $p \geq 1$) , then*

$$H^q(Y, S) \cong \Gamma(B, f^q_* S). \tag{0.5}$$

0.2 Twistor correspondences

Holomorphic twistor correspondence

Definition 0.14 *Let Z, F and X be complex manifolds. A diagram*

$$\begin{tikzcd} & F \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ Z & & X \end{tikzcd}$$

is said to be a <u>holomorphic correspondence of complex manifolds</u> if μ, ν are surjective holomorphic maps of maximal rank and $\mu \times \nu : F \to X \times Z$ is an embedding.

Definition 0.15 *A <u>holomorphic twistor correspondence</u> is a holomorphic correspondence of complex manifolds with the additional condition that the map ν is proper.*

The space Z will be referred to as a *twistor space* for X, and F as the *correspondence space* for the twistor correspondence.

Most examples of holomorphic twistor correspondence in fact come from the following construction.

Theorem 0.16 (Kodaira[21]) *Let Z be a complex manifold, Y a compact complex submanifold of Z and N the normal bundle of Y in Z. If $H^1(Y, N) = 0$,* *then Y belongs to a locally complete family of compact complex submanifolds $\{Y_x : x \in X\}$, for some complex manifold X, and there is a canonical isomorphism $T_xX \cong H^0(Y_x, N_x)$, where N_x is the normal bundle of $Y_x \subset Z$.*

Definition 0.17 *A pair $(z, x) \in Z \times X$ is said to be incident if z lies in the complex compact submanifold associated with x.*

To relate Z and X, one introduces

$$F := \{(z, x) \in Z \times X \mid (z, x) \text{ is incident }\}.$$

Then we obtain a holomorphic twistor correspondence, where μ, ν are the obvious forgetful maps.

Remark: In general, the infinitesimal intersection properties of the Y_x endow X with some differential geometric structure. See [16], [28] for examples.

Non-holomorphic twistor correspondence

Definition 0.18 *A correspondence of smooth manifolds is a diagram*

$$\begin{tikzcd} & F_R \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z_R & & X_R, \end{tikzcd}$$

where Z_R, F_R and X_R are smooth manifolds, η, τ are submersions and the map $\eta \times \tau : F_R \to Z_R \times X_R$ is an embedding.

Definition 0.19 *A non-holomorphic twistor correspondence is a correspondence of smooth manifolds with the additional condition that Z_R is a complex manifold and that $\eta(\tau^{-1}(x))$ is a complex compact submanifold of Z_R, for each $x \in X$.*

Again Z_R is a *twistor space* for X_R, and F_R is the *correspondence space* for this twistor correspondence.

0.3 Homogeneous vector bundles

Let $M = G/H$ be a homogeneous space, where G is a Lie group and H is a closed subgroup of G. Let V_0 be a finite dimensional H-module and $\rho : H \longrightarrow GL(V_0)$ be the corresponding representation of H. Then one can define a vector bundle on M as follows:

Introduce an equivalence relation $\sim$ on the product $G \times V_0$ by setting

$$(g, v) \sim (gh, \rho(h)^{-1}v) \quad \text{for } h \in H.$$

Let $G \times_H V_0$ denote the set of equivalence classes in $G \times V_0$ under $\sim$. Then there is a natural mapping

$$\begin{array}{ccc} G \times_H V_0 & \xrightarrow{\pi} & G/H \\ [(g, v)] & \longmapsto & gH, \end{array}$$

where the square bracket indicates taking the equivalence class. One can check that this indeed gives rise to a vector bundle on M (cf. [35]).

Definition 0.20 *The vector bundle $V := G \times_H V_0$ is called the* <u>homogeneous vector bundle</u> *associated to the H-module V_0.*

We shall use the notation

$$V \longleftrightarrow V_0$$

to denote the 1-1 correspondence between a homogeneous vector bundle and its associated H-module. In particular, the tangent bundle and complexified tangent bundle on G/H are homogeneous vector bundles associated to adjoint representations of H on $\mathfrak{g}_0/\mathfrak{h}_0$ and $\mathfrak{g}/\mathfrak{h}$ respectively, cf. [20], and we write

$$T(G/H)^{\mathbb{R}} \longleftrightarrow \mathfrak{g}_0/\mathfrak{h}_0, \quad T(G/H)^{\mathbb{C}} \longleftrightarrow \mathfrak{g}/\mathfrak{h}.$$

Note: For a real Lie group G, we use $\mathfrak{g}_0$ to denote its Lie algebra, and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. However, for a complex Lie group G, we will simply use $\mathfrak{g}$ to denote its Lie algebra.

Group action on $G \times_H V_0$ and $\mathcal{E}(G \times_H V_0)$

The group action of G on M lifts to an action of G on $G \times_H V_0$ and $\mathcal{E}(G \times_H V_0)$, given by

$$g'[g, v] = [g'g, v], \qquad \text{and} \qquad g's(gH) = s(g'^{-1}gH),$$

respectively, where $g' \in G$ and $s : M \longrightarrow G \times_H V_0$ is a smooth section of $G \times_H V_0$.

To any section s of $G \times_H V_0$, one can associate a smooth function $f_s : G \longrightarrow V_0$ by

$$s(gH) = [g, f_s(g)] \in G \times_H V_0,$$

with $f_s(gh) = \rho(h)^{-1} f_s(g)$. Then one obtains

$$\mathcal{E}(G \times_H V_0) \cong \{f_s : G \longrightarrow V_0 \mid f_s \text{ smooth},\ f_s(gh) = \rho(h)^{-1} f_s(g),\ h \in H, g \in G\}.$$

The group action of G on f_s is given by

$$g' f_s(g) = f_s(g'^{-1} g).$$

0.4 Conformal structure and projective structure

Definition 0.21 *Two complex Riemannian metrics g_{ab} and h_{ab} on a complex manifold M are said to be conformally equivalent if they are related by $g_{ab} = \Omega^2 h_{ab}$, where Ω is a nonvanishing holomorphic function on M. A conformal structure on M is a conformally equivalent class of metrics.*

Definition 0.22 *A conformally weighted function f of weight k on M is an association of a function to each metric in the conformal class in such a way that under conformal rescaling $g_{ab} \longrightarrow \Omega^2 g_{ab}$, we have $f \longrightarrow \Omega^k f$.*

Definition 0.23 *Two torsion-free affine connections ∇_a and $\widetilde{\nabla}_a$ on a manifold M are said to be projectively equivalent if they have the same geodesics, considered as unparameterized paths. A projective structure on M is defined as such an equivalence class of affine connections.*

Lemma 0.24 *Concretely, two torsion-free affine connections ∇_a, $\tilde{\nabla}_a$ are projectively equivalent if they are related by*

$$\tilde{\nabla}_a U^b = \nabla_a U^b + \gamma_a U^b + \gamma_c U^c \delta_a{}^b,$$

where the 1-form γ_a is arbitrary.

Definition 0.25 *A conformal structure $[g_{ab}]$ on a complex 3-manifold is said to be compatible with a projective structure $[\nabla_a]$ if*

$$\nabla_a g_{bc} = \alpha_a g_{bc} + \beta_b g_{ac} + \beta_c g_{ab} + \gamma_{abc},$$

where $\gamma_{abc} + \gamma_{cab} + \gamma_{bca} = 0$ and $\gamma_{abc} = \gamma_{acb}$.

Geometrically this means every geodesic (with respect to $[\nabla_a]$) starting off in a null direction (with respect to $[g_{ab}]$) is a null curve, cf. [16].

Lemma 0.26 *Given a compatible pair $([\nabla_a], [g_{ab}])$, there is a distinguished connection $\nabla_a \in [\nabla_a]$, s.t.*

$$\nabla_a g_{bc} = \alpha_a g_{bc}$$

for some 1-form α.

Proof: See [16]. □

0.5 Spinor index notation

We shall adopt Penrose's abstract index notation, see [29] for details, where indices are used as 'place markers', indicating the space to which a tensor belongs, with no implication of a choice of basis.

If S is an $SL(2,\mathbb{C})$-bundle, which we shall denote by S^A, then there exists a nondegenerate skew form

$$\epsilon \in \Gamma(M, \wedge^2 S).$$

We can then write the ϵ as $\epsilon^{AB} \in S^{[AB]} \cong \wedge^2 S$. Note we shall often abuse notation by denoting both a vector bundle S and the sheaf of germs of sections of S by S. The skew form ϵ^{AB} gives an isomorphism from S_A, the dual of S^A, to S^A, which can be written explicitly as

$$\xi_A \longmapsto \xi^A := \epsilon^{AB}\xi_B.$$

Similarly one has $\epsilon_{AB} \in S_{[AB]}$ which maps S^A isomorphically to S_A via

$$\xi^a \longmapsto \xi_B := \xi^A\epsilon_{AB}.$$

Spinor index can be understood numerically also. Choosing a frame (e_0^A, e_1^A) for S^A and letting (e_A^0, e_A^1) be the dual frame, one can express a section σ of S^A as

$$\sigma = \sigma^0 e_0^A + \sigma^1 e_1^A = \sigma^{\mathbf{A}} e_{\mathbf{A}}^A,$$

using the summation convention, where $\mathbf{A} = 0, 1$. Then $\sigma^{\mathbf{A}}$ is numerically indexed and it represents σ with respect to the frame (e_0^A, e_1^A). Notice the difference between indices '$\mathbf{A}$' and 'A'. We will usually make a choice of frame, called *spinor frame*, such that

$$\epsilon = \epsilon^{AB} \sim \epsilon^{\mathbf{AB}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the dual frame will give

$$\epsilon^* = \epsilon_{AB} \sim \epsilon_{\mathbf{AB}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

0.6 Some results on direct images

Definition 0.27 *Let B be a complex manifold with a holomorphic rank two bundle $\mathcal{O}^A$. The bundle $\mathcal{O}[n]$, $n \in \mathbb{Z}$, is defined as follows*

$$\begin{aligned} \mathcal{O}[1] &:= \mathcal{O}^{[AB]}, & \mathcal{O}[-1] &:= \mathcal{O}_{[AB]} \cong \mathcal{O}[1]^*, \\ \mathcal{O}[m] &:= \otimes^m \mathcal{O}[1], & \mathcal{O}[-m] &:= \otimes^m \mathcal{O}[-1], \qquad m \in \mathbb{N}, \\ \mathcal{O}[0] &:= \mathcal{O}. \end{aligned}$$

Similarly, for $\mathcal{E}^A$, a smooth complex rank two bundle on a smooth manifold B, one can define $\mathcal{E}[1] := \mathcal{E}^{[AB]}$, etc. for the smooth category.

Definition 0.28 *Let B be a complex manifold with a holomorphic rank 2 bundle $\mathcal{O}_A$, Y be the total space of $\mathbb{P}(\mathcal{O}_A)$, and $\nu : Y \longrightarrow B$ be the projection. A point of Y can be written as (b, z), where $b \in B$ and $z \in \nu^{-1}(b)$. Define $\mathcal{O}(n)$ on Y to be the line bundle given by*

$$\mathcal{O}(n)_{(b,z)} = (\mathcal{O}_{\nu^{-1}(b)}(n))_z,$$

where $\mathcal{O}_{\nu^{-1}(b)}(n)$ is the usual $\mathcal{O}(n)$ bundle on $\nu^{-1}(b)$, which by definition is biholomorphic to $\mathbb{CP}_1$.

Proposition 0.29 *Let $\nu : Y \longrightarrow B$ be as above, we have*

$$\begin{cases} \nu_*\mathcal{O}(n) \cong \mathcal{O}^{\overbrace{(AB\ldots D)}^{n}} & \text{for } n \geq 0 \\ \nu^1_*\mathcal{O}(-n-2) \cong \mathcal{O}_{\underbrace{(AB\ldots D)}_{n}}[-1] & \text{for } n \geq 0 \\ \text{vanishes otherwise.} & \end{cases}$$

Proof: The proof given in [8] works for the general case here also: one just replaces the bundles $\mathcal{O}^{A'}$, $\mathcal{O}_{A'}$ there with the bundles $\mathcal{O}^A$, $\mathcal{O}_A$ here, and makes the subsequent changes. □

Definition 0.30 *Let B be a smooth manifold with a smooth rank 2 bundle $\mathcal{E}_A$, Y be the total space of $\mathbb{P}(\mathcal{E}_A)$, and $\tau : Y \longrightarrow B$ be the projection. A point of Y can be written as (b, z), where $b \in B$ and $z \in \tau^{-1}(b)$. Define $\text{Œ}(n)$ on Y to be the line bundle given by*

$$\text{Œ}(n)_{(b,z)} = (\mathcal{O}_{\tau^{-1}(b)}(n))_z,$$

where $\mathcal{O}_{\tau^{-1}(b)}(n)$ is the usual $O(n)$ bundle on $\tau^{-1}(b) \cong \mathbb{CP}_1$.

Note: The smooth line bundle $\text{Œ}(n)$ is holomorphic when restricted to fibres of τ. We shall also use $\text{Œ}(n)$ to denote the sheaf of germs of smooth sections of the bundle $\text{Œ}(n)$ which are holomorphic on fibres of τ.

Proposition 0.31 *Let $\tau : Y \longrightarrow B$ be a submersion of smooth manifolds as above, then*

$$\begin{cases} \tau_* Œ(n) \cong \mathcal{E}^{\overbrace{(AB\ldots D)}^{n}} & \textit{for } n \geq 0 \\ \tau_*^1 Œ(-n-2) \cong \mathcal{E}_{\underbrace{(AB\ldots D)}_{n}}[-1] & \textit{for } n \geq 0 \\ \textit{vanishes otherwise.} & \end{cases}$$

Proof: The proof is essentially the same as that for the holomorphic case, except that here we have

$$H^q(\tau^{-1}(U), Œ(n)) \cong \{\text{smooth } g : U \longrightarrow H^q(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(n))\},$$

for U an open subset of B. □

Remark: When the bundle $\mathcal{O}^A$ is reduced to an $SL(2,\mathbb{C})$-bundle, then $\wedge^2\mathcal{O}^A = \mathcal{O}[1]$ is trivial and one can just drop the numbers in square brackets.

Chapter 1

The Holomorphic Penrose Transform

We start by defining a general Penrose transform in the holomorphic category, essentially following [8]. The holomorphic Penrose transform, cf. [4,8,9], is a mechanism which translates holomorphic data on one complex manifold Z to that on another, X, where Z and X are related in the following way:

Condition 1: There exists a *holomorphic correspondence of complex manifolds*, cf. Definition 0.14,

$$\begin{array}{ccccc} & & F & & \\ & \overset{\mu}{\swarrow} & & \overset{\nu}{\searrow} & \\ Z & & & & X, \end{array} \tag{1.1}$$

where, in general, X will be assumed to be Stein.

Condition 2: Each fibre of μ is contractible.

Condition 3: The map ν is proper.

An extra **Condition 4** will be introduced in section 1.4.

Pull-back mechanism

Lemma 1.1 *Let $\mathcal{S}$ be a sheaf on Z, then there exists a canonical homomorphism*

$$H^p(Z, \mathcal{S}) \xrightarrow{\mu^*} H^p(F, \mu^{-1}\mathcal{S}).$$

Proof: see [35]. □

Theorem 1.2 (Buchdahl) *Let Z, F be complex manifolds, $\mu : F \to Z$ be a surjective holomorphic mapping of maximal rank, and V be a holomorphic vector bundle over Z. If for some $N \geq 0$, $H^p(\mu^{-1}(x), \mathbb{C}) = 0$ for $p = 0, 1, \ldots, N$ and for all $x \in Z$, then the canonical homomorphism*

$$H^q(Z, \mathcal{O}(V)) \to H^q(F, \mu^{-1}\mathcal{O}(V))$$

is an isomorphism for $q = 0, 1, \ldots, N$ and a monomorphism for $q = N + 1$.

Proof: see [7]. □

Condition 2 then immediately gives us

$$H^q(Z, \mathcal{O}(V)) \cong H^q(F, \mu^{-1}\mathcal{O}(V)) \quad \text{for all } q. \tag{1.2}$$

Remark: If the fibres of μ are not contractible, one can still proceed, although it is more complicated, cf. 4.2.1 of [1] and the references therein. However, for simplicity, we will assume condition 2.

Spectral sequence

As the twisted holomorphic relative de Rham sequence on F

$$0 \to \mu^{-1}\mathcal{O}_Z(V) \longrightarrow \mathcal{O}_F(\mu^*V) \xrightarrow{d_\mu} \Omega^1_\mu(\mu^*V) \to \ldots \to \Omega^k_\mu(\mu^*V) \to 0$$

is exact (Lemma 0.3), there exists a spectral sequence associated with the resolution (Theorem 0.9)

$$E_1^{p,q} = H^q(F, \Omega^p_\mu(\mu^*V)) \Longrightarrow H^r(F, \mu^{-1}\mathcal{O}_Z(V)). \tag{1.3}$$

Push-down mechanism

By Theorem 0.12 there exists a spectral sequence

$$E_2^{p,q} = H^p(X, \nu^q_*\mathcal{S}) \Longrightarrow H^r(F, \mathcal{S}),$$

for any sheaf $\mathcal{S}$ on F. As X is Stein, $H^p(X, \nu_*^q \mathcal{S}) = 0$ for $p \geq 1$. Therefore, letting $\mathcal{S} = \Omega_\mu^s(\mu^* V)$, $s \geq 0$, one obtains an isomorphism (Corollary 0.13)

$$H^q(F, \Omega_\mu^s(\mu^* V)) \cong \Gamma(X, \nu_*^q(\Omega_\mu^s(\mu^* V))). \tag{1.4}$$

To avoid dimension jumps of $H^q(\nu^{-1}(x), \Omega_\mu^s(\mu^* V))$ at some exceptional points $x \in X$, we shall in addition assume

Condition 4: The dimension of $H^q(\nu^{-1}(x), \Omega_\mu^s(\mu^* V))$ remains constant as x varies.

In this case, the sheaves $\nu_*^q(\Omega_\mu^s(\mu^* V))$ can be regarded as sheaves of germs of holomorphic sections of holomorphic vector bundles over X.

Summary

Theorem 1.3 *Given a correspondence (1.1) satisfying the conditions 1-4 of this chapter, and a holomorphic vector bundle $V \longrightarrow Z$, there is a spectral sequence*

$$E_1^{p,q} = \Gamma(X, \nu_*^q(\Omega_\mu^p(\mu^* V))) \Longrightarrow H^{p+q}(Z, \mathcal{O}(V)). \tag{1.5}$$

We refer to this as the holomorphic Penrose transform.

Chapter 2

A Holomorphic Penrose Transform for Euclidean 3-space

In this chapter, we apply the methods of Chapter 1 to an example where the target space X is complex Euclidean 3-space $\mathbb{C}^3$, considered as the space of 2×2 trace-free complex matrices. The twistor correspondence we study is well-known [18,19]. If one regards the twistor space as a homogeneous space for the group generated by $SL(2, \mathbb{C})$ and translations, then the homogeneous line bundles are classified by one integer n and one complex parameter λ. In [18], the Penrose transform for the case where $\lambda = 0$ is studied, cf. the Introduction chapter. We extend this to all values of the parameter λ. In particular, we describe homogeneous line bundles whose Penrose transform yields eigenfunctions of the Laplacian with non-zero eigenvalues.

Section 2.1 is the basic set up and Section 2.2 gives the results. Section 2.3 is a discussion of some degeneracy phenomena for the case $\lambda = 0$. In an Appendix, an explicit contour integral formula is given for eigenfunctions of the Laplacian.

2.1 Correspondence and set-up

Consider $\mathbb{C}^3$ with the standard symmetric $\mathbb{C}$-bilinear form. Let $\mathbb{CP}_1 \hookrightarrow \mathbb{CP}_2$ be the embedding of the space of null directions in the projective tangent space at $\underline{0} \in \mathbb{C}^3$. If $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ are homogeneous coordinates on $\mathbb{CP}_1$, this embedding can be given by

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}\frac{1}{2}(\alpha^2-\beta^2)\\ \frac{i}{2}(\alpha^2+\beta^2)\\ -\alpha\beta\end{pmatrix}.$$

A general null hyperplane in $\mathbb{C}^3$ is given by $\underline{n}\cdot\underline{x}=\xi$, where $\underline{n}$ is a null vector, $\xi\in\mathbb{C}$, up to the equivalence $(\underline{n},\xi)\sim(\lambda\underline{n},\lambda\xi)$.

Thus one can identify the space T of null hyperplanes with

$$\left\{\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\xi\right)\right\}/\sim,$$

where $\left(\binom{\alpha}{\beta},\xi\right)\sim\left(\binom{\lambda\alpha}{\lambda\beta},\lambda^2\xi\right)$. There is a natural map $T\to\mathbb{CP}_1$, and in fact, T is the total space of the line bundle $\mathcal{O}(2)$.

Next, we define the correspondence space $\mathbb{F}$ by

$$\mathbb{F}:=\{(p,\alpha)\mid p\in\mathbb{C}^3,\alpha \text{ is a null plane in } \mathbb{C}^3, p\in\alpha\}.$$

From the above, $\mathbb{F}$ is actually isomorphic to $\mathbb{C}^3\times\mathbb{P}_1$ and one has:

$$\begin{tikzcd} & \mathbb{C}^3\times\mathbb{P}_1 \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ T & & \mathbb{C}^3, \end{tikzcd}$$

where μ and ν are the obvious forgetful maps. For $p\in T$, $\nu\circ\mu^{-1}$ is a null plane in $\mathbb{C}^3$, and the fibre of ν at $x\in\mathbb{C}^3$ is biholomorphic to $\mathbb{CP}_1$, corresponding to the space of null directions at x, whose image under μ is a section of $\mathcal{O}(2)$. We will call T the *minitwistor space of* $\mathbb{C}^3$, [19].

Coordinates and group actions

Represent $\mathbb{C}^3$ as $\{2\times 2$ tracefree complex matrices$\}$,

$$\mathbb{C}^3\ni(x,y,z)\longmapsto X=\begin{pmatrix} z & x-iy\\ x+iy & -z\end{pmatrix},$$

with the bilinear form

$$\langle X, X' \rangle = \frac{1}{2}\text{trace}(XX').$$

Next, we introduce coordinates on $\mathbb{F}$: $\left(X, \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\right) \in \mathbb{C}^3 \times \mathbb{P}_1$. Then the maps μ, ν can be written explicitly as

$$\begin{aligned}
\mu : \mathbb{C}^3 \times \mathbb{CP}_1 &\to T; \quad \left(X, \begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right) \mapsto \left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, (-\beta \quad \alpha) X \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right)\\
\nu : \mathbb{C}^3 \times \mathbb{CP}_1 &\to \mathbb{C}^3; \quad \left(X, \begin{bmatrix}\alpha\\ \beta\end{bmatrix}\right) \mapsto X.
\end{aligned}$$

Definition 2.1 *The group $ESL(2,\mathbb{C})$ is defined as follows,*

$$ESL(2,\mathbb{C}) = \{(A, B) \mid A \in SL(2,\mathbb{C}), B \in \{2 \times 2 \text{ complex tracefree matrices}\}\},$$

where the group operation is: $(A, B) \circ (A', B') = (AA', AB'A^{-1} + B)$. The multiplication and addition on the right hand side are just the usual matrix multiplication and addition.

The transitive group actions of $ESL(2,\mathbb{C})$ on the spaces are

$$\begin{aligned}
&(A, B)X \longmapsto AXA^{-1} + B && \text{on } \mathbb{C}^3,\\
&(A, B)\left(X, \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\right) \longmapsto \left(AXA^{-1} + B, \left[A\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)\right]\right) && \text{on } \mathbb{F},\\
&(A, B)(\mathcal{U}, \xi)/\!\sim\; = (A\mathcal{U}, \xi + \widetilde{\mathcal{U}}A^{-1}BA\mathcal{U})/\!\sim && \text{on } T,
\end{aligned}$$

where $\mathcal{U}$, $\widetilde{\mathcal{U}}$ stand for $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and $(\,-\beta \quad \alpha\,)$ respectively.

In fact $ESL(2,\mathbb{C})$ is the universal covering group of the complex Euclidean group on $\mathbb{C}^3$, where $(A, B) \in ESL(2,\mathbb{C})$ corresponds to an orthogonal transformation A followed by a translation B, and the differential $d\phi_g$ of the homeomorphism $\phi_g : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ induced by $g = (A, B)$ preserves $\langle\ ,\ \rangle$, i.e. $\langle X, X' \rangle_{x_0} = \langle d\phi_g X, d\phi_g X' \rangle_{\phi_g(x_0)}$, $\forall X, X' \in T_{x_0}\mathbb{C}^3$.

Homogeneous vector bundles

Definition 2.2 *Define $\mathcal{O}^A$ to be the homogeneous vector bundle on $\mathbb{C}^3$ associated to the standard representation of $SL(2,\mathbb{C})$ (the isotropy group at $\underline{0} \in \mathbb{C}^3$). The vector bundle $\mathcal{O}_A$ is its dual.*

We can identify $T\mathbb{C}^3$ with $(\mathcal{O}^A \otimes \mathcal{O}_B)_\circ$, the tracefree part of $\mathcal{O}^A \otimes \mathcal{O}_B$. Also, as $\mathcal{O}^A$ is an $SL(2,C)$-bundle, there exists a nondegenerate skew form $\epsilon^{AB} \in \Gamma(\mathbb{C}^3, \Lambda^2\mathcal{O}^A)$, which provides an isomorphism between $\mathcal{O}_A$ and $\mathcal{O}^A$.

For T, one has $T = ESL(2,\mathbb{C})/Q$, where

$$Q = \left\{ \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} u & w \\ 0 & -u \end{pmatrix} \right) \quad a,b,u,w \in \mathbb{C},\ a \neq 0 \right\}.$$

Definition 2.3 *Define a line bundle $\mathcal{O}(n,\lambda)$ on T to be the homogeneous holomorphic vector bundle associated to the representation of Q given by*

$$\rho\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} u & w \\ 0 & -u \end{pmatrix} \right) = a^{-n}e^{-\lambda u}, \quad n \in \mathbb{Z},\ \lambda \in \mathbb{C}.$$

We shall write $\mathcal{O}_T(n)$ for $\mathcal{O}(n,0)$, and $\mathcal{O}_{\mathbb{F}}(n)$ for $\mu^*\mathcal{O}_T(n)$. We shall also abuse notation by denoting both the bundle $\mathcal{O}^A$ on $\mathbb{C}^3$ and its pullback $\nu^*\mathcal{O}^A$ on $\mathbb{F}$ by $\mathcal{O}^A$.

As null planes of $\mathbb{C}^3$ are contractible, by Theorem 1.2 one has

$$H^p(T,\mathcal{O}(n,\lambda)) \cong H^p(\mathbb{F},\mu^{-1}\mathcal{O}(n,\lambda)), \quad p = 0,1. \tag{2.1}$$

The relative de Rham resolution

Lemma 2.4 *One has the following canonical isomorphisms on $\mathbb{F}$,*

$$\Omega^1_\mu \cong \mathcal{O}_A(1), \quad \Omega^2_\mu \cong \mathcal{O}(2).$$

Proof: Let R be the isotropy group for $\mathbb{F}$ at $(\underline{0}, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$. The relative 1-form Ω^1_μ is associated to the R-module $(\mathfrak{q}/\mathfrak{r})^*$, because it is defined by (0.1) which is associated to the s.e.s. of R-modules

$$0 \longrightarrow (\mathfrak{g}/\mathfrak{q})^* \longrightarrow (\mathfrak{g}/\mathfrak{r})^* \longrightarrow (\mathfrak{q}/\mathfrak{r})^* \longrightarrow 0.$$

Similarly, one has $\Omega^2_\mu \longleftrightarrow \Lambda^2(\mathfrak{q}/\mathfrak{r})^*$.

Consider the Adjoint action of R on $\mathfrak{q}/\mathfrak{r}$ first: For $r = \left(\left(\begin{smallmatrix}\alpha & \beta \\ 0 & \alpha^{-1}\end{smallmatrix}\right), 0\right) \in R$, $Y = \left(0, \left(\begin{smallmatrix}z & w \\ 0 & -z\end{smallmatrix}\right)\right) \in \mathfrak{q}/\mathfrak{r}$, we have

$$\mathrm{Ad}(r)Y = \left(0, \begin{pmatrix} z & \alpha^2 w - 2\alpha\beta z \\ 0 & -z \end{pmatrix}\right).$$

Now write Y as a column vector $\begin{pmatrix} w \\ -2z \end{pmatrix} \in \mathbb{C}^2$. Then the Adjoint representation of R is simply the following

$$\mathrm{Ad}(r)\begin{pmatrix} w \\ -2z \end{pmatrix} = \alpha\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} w \\ -2z \end{pmatrix} = \begin{pmatrix} \alpha^2 w - 2\alpha\beta z \\ -2z \end{pmatrix}.$$

Thus $\Omega^{1^*}_\mu$ is associated to the bundle $\mu^*\mathcal{O}(-1)\otimes\nu^*\mathcal{O}^A$, which we denote by $\mathcal{O}^A(-1)$. Taking its dual, we then obtain $\Omega^1_\mu \cong \mathcal{O}_A(1)$.

For Ω^2_μ, we have

$$\Omega^2_\mu \cong \wedge^2\mathcal{O}_A(1) \cong \mathcal{O}_{[AB]}(2) \cong \mathcal{O}(2),$$

as $\mathcal{O}_{[AB]}$ is canonically trivial. □

Proposition 2.5 *The relative de Rham resolution is*

$$0 \longrightarrow \mu^{-1}\mathcal{O}_T \longrightarrow \mathcal{O} \overset{\pi_B\nabla_A{}^B}{\longrightarrow} \mathcal{O}_A(1) \overset{\pi_C\nabla^{CA}}{\longrightarrow} \mathcal{O}(2) \longrightarrow 0, \tag{2.2}$$

where $\mathcal{O}$ without subscript stands for $\mathcal{O}_{\mathbb{F}}$, and $\nabla_A{}^B$ is the spinor version of the Levi-Civita connection, lifted to $\mathbb{F}$.

Proof: Write

$$X^{\mathbf{A}}{}_{\mathbf{B}} := \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \in \mathbb{C}^3, \quad \text{and } \pi_{\mathbf{A}} := [\pi_0, \pi_1].$$

Introduce local affine coordinate

on $\mathbb{F}_0 = \mathbb{C}^3 \times (\mathbb{P}_1\backslash\{\pi_0 = 0\})$:

$$s^0 = 2z + (x+iy)\tfrac{\pi_1}{\pi_0} \qquad s^1 = x+iy$$

$$q = -(x-iy) + 2z(\tfrac{\pi_1}{\pi_0}) + (x+iy)\left(\tfrac{\pi_1}{\pi_0}\right)^2 \qquad r = \tfrac{\pi_1}{\pi_0},$$

on $T_0 = \{(\pi_{\mathbf{A}},\xi)/\sim, \pi_0 \neq 0\}$:

$$q = \tfrac{\xi}{(\pi_0)^2} \qquad r = \tfrac{\pi_1}{\pi_0}.$$

By restricting to $\mathbb{F}_0$, the map μ becomes $(r,q,s^0,s^1) \longmapsto (r,q)$, and one has

$$d_\mu f = \frac{\partial f}{\partial s^{\mathbf{A}}} d_\mu s^{\mathbf{A}}.$$

Now write $\nabla_A{}^B$ explicitly as

$$\nabla_{\mathbf{A}}{}^{\mathbf{B}} = 2\begin{pmatrix} \frac{1}{2}\frac{\partial}{\partial z} & \frac{\partial}{\partial(x-iy)} \\ \frac{\partial}{\partial(x+iy)} & \frac{-1}{2}\frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x}+i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x}-i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{pmatrix},$$

and observe

$$\begin{aligned} 2\pi_0\frac{\partial}{\partial s^0} &= 2\pi_0\left(r\frac{\partial}{\partial(x-iy)}+\frac{1}{2}\frac{\partial}{\partial z}\right) &= (\pi_1\nabla_0{}^1+\pi_0\nabla_0{}^0) &= \pi_{\mathbf{B}}\nabla_0{}^{\mathbf{B}} \\ 2\pi_0\frac{\partial}{\partial s^1} &= 2\pi_0\left(\frac{\partial}{\partial(x+iy)}-\frac{r}{2}\frac{\partial}{\partial z}\right) &= (\pi_0\nabla_1{}^0+\pi_1\nabla_1{}^1) &= \pi_{\mathbf{B}}\nabla_1{}^{\mathbf{B}}. \end{aligned}$$

That is, $\pi_B\nabla_A{}^B f$ agrees with $2\frac{\partial f}{\partial s^{\mathbf{A}}}$ in this affine coordinate. As $\Omega^1_\mu \cong \mathcal{O}_A(1)$, the first d_μ can be written as $\pi_B\nabla_A{}^B$. Note π_A is just the composition of $\pi^A : \mathcal{O}(-1) \longrightarrow \mathcal{O}^A$ and $\epsilon_{AB} : \mathcal{O}^A \longrightarrow \mathcal{O}_B$, and ∇_A^B makes sense on $\mathbb{F} = \mathbb{C}^3 \times \mathbb{P}_1$ by acting on $\mathbb{C}^3$ alone.

The second d_μ can be worked out similarly. □

Proposition 2.6 *The twisted relative de Rham sequence is*

$$0 \longrightarrow \mu^{-1}\mathcal{O}(n,\lambda) \xrightarrow{i} \mathcal{O}(n) \xrightarrow{\pi_B\nabla_A{}^B+\lambda\pi_A} \mathcal{O}_A(n+1) \xrightarrow{\pi_C\nabla^{AC}+\lambda\pi^A} \mathcal{O}(n+2) \longrightarrow 0.$$

Proof: Tensoring (2.2) with $\mu^{-1}\mathcal{O}(n,\lambda)$, we obtain

$$0 \longrightarrow \mu^{-1}\mathcal{O}(n,\lambda) \xrightarrow{i} \mathcal{O}(n) \xrightarrow{d_\mu} \mathcal{O}_A(n+1) \xrightarrow{d_\mu} \mathcal{O}(n+2) \longrightarrow 0,$$

where we have used $\mu^{-1}\mathcal{O}(n,\lambda) \otimes_{\mu^{-1}\mathcal{O}_T} \mathcal{O}_{\mathbb{F}} \cong \mu^*\mathcal{O}(n,\lambda) \cong \mathcal{O}_{\mathbb{F}}(n)$.

As the inclusion i from $\mu^{-1}\mathcal{O}(n,\lambda)$ to $\mu^*\mathcal{O}(n,\lambda)$ depends on λ, the differential operator from $\mathcal{O}(n)$ to $\mathcal{O}_A(n+1)$ annihilating sections of $\mu^{-1}\mathcal{O}(n,\lambda)$ varies accordingly. By representing sections of $\mathcal{O}(n,\lambda)$ as holomorphic functions h described as in Appendix, one can do explicit calculation on $h(\pi^A, X^A{}_B\pi^B)$. It is then straightforward to check that the first d_μ is $\pi_B\nabla_A{}^B+\lambda\pi_A$, as the latter is precisely the operator that annihilates such h. The second d_μ can then be identified with $\pi_C\nabla^{AC}+\lambda\pi^A$ by $d_\mu \cdot d_\mu = 0$. □

Direct images

Lemma 2.7 *For $n \geq 0$,*

$$\nu_*\mathcal{O}(n) \xrightarrow{\nu_*(\pi_B\nabla_A{}^B+\lambda\pi_A)} \nu_*\mathcal{O}_A(n+1) \xrightarrow{\nu_*(\pi_C\nabla^{AC}+\lambda\pi^A)} \nu_*\mathcal{O}(n+2)$$

can be identified with

$$\begin{array}{ccccc}
\mathcal{O}^{\overbrace{(C\ldots E)}^{n}} & \longrightarrow & \mathcal{O}_A^{(BC\ldots E)} & \longrightarrow & \mathcal{O}^{(HB\ldots E)} \\
\Cup & & \Cup & & \\
\phi^{C\ldots E} & \longmapsto & \nabla_A^{(B}\phi^{C\ldots E)} + \lambda\epsilon_A^{(B}\phi^{C\ldots E)} & & \Cup \\
 & & \psi_A{}^{B\ldots E} & \longmapsto & \nabla^{A(H}\psi_A{}^{B\ldots E)} + \lambda\epsilon^{A(H}\psi_A{}^{B\ldots E)}.
\end{array}$$

Proof: For direct images, use Prop 0.29, (use $\mathcal{O}_A \cong \mathcal{O}^A$ also), while the subscript 'A' here is just a passenger. For the first map, representing elements in $\mathcal{O}(n)$ by $\pi_C \ldots \pi_E \phi^{C\ldots E}$, where $\phi^{C\ldots E}$ is a local section of $\odot^n \mathcal{O}^A$, then the map $\pi_B \nabla_A^{\ B} + \lambda\pi_A :$ $\mathcal{O}(n) \to \mathcal{O}_A(n+1)$ gives

$$\begin{aligned}
\pi_C \ldots \pi_E \phi^{\overbrace{C\ldots E}^{n}} &\longmapsto (\pi_B \nabla_A^{\ B} + \lambda\pi_A)(\pi_C \ldots \pi_E \phi^{C\ldots E}) \\
&= \pi_B \ldots \pi_E (\nabla_A^{\ B} + \lambda\epsilon_A^{\ B})\phi^{C\ldots E}.
\end{aligned}$$

The second map can be identified by similar means. □

Lemma 2.8 *For $n \geq 1$, one has the following*

$$\begin{array}{ccc}
\nu_*^1 \mathcal{O}(-n-2) & \xrightarrow{\nu_*^1(\pi_B \nabla_A^{\ B} + \lambda\pi_A)} & \nu_*^1 \mathcal{O}_A(-n-1) \\
\wr\| & & \wr\| \\
\mathcal{O}_{\underbrace{(B\ldots E)}_{n}} & \xrightarrow{\nabla_A^{\ B} + \lambda\epsilon_A^{\ B}} & \mathcal{O}_{A(C\ldots E)} \\
\Cup & & \Cup \\
\phi_{B\ldots E} & \longmapsto & \nabla_A^{\ B}\phi_{B\ldots E} + \lambda\phi_{A\ldots E}\,.
\end{array}$$

Proof: Again, use Prop 0.29 to get direct images. It is straightforward to check that the following diagram commutes, and has exact rows.

$$\begin{array}{ccccccccc}
0 \to & \mathcal{O}(-n-2) & \xrightarrow{\overbrace{\pi_B \ldots \pi_F}^{n+1}} & \mathcal{O}_{(B\ldots F)}(-1) & \xrightarrow{\pi^F} & \mathcal{O}_{(B\ldots E)} & \to 0 \\
 & \downarrow{\scriptstyle \pi_B \nabla_A^{\ B} + \lambda\pi_A} & & \downarrow{\scriptstyle \nabla_A^{\ B} + \lambda\epsilon_A^{\ B}} & & \downarrow{\scriptstyle \nabla_A^{\ B} + \lambda\epsilon_A^{\ B}} & \\
0 \to & \mathcal{O}_A(-n-1) & \xrightarrow{\pi_C \ldots \pi_F} & \mathcal{O}_{A(C\ldots F)}(-1) & \xrightarrow{\pi^F} & \mathcal{O}_{A(C\ldots E)} & \to 0.
\end{array}$$

The result then follows by taking direct images. □

2.2 Results

We will discuss the Penrose transform for $H^0(T, \mathcal{O}(n,\lambda))$ first, then $H^1(T, \mathcal{O}(n,\lambda))$ in three cases: (i) $n \le -3$ (ii) $n \ge -1$ (iii) $n = -2$, using the spectral sequence (1.5) and Lemma 2.7, 2.8.

Zeroth cohomology:

Proposition 2.9 *When $n < 0$ or $\lambda \neq 0$, $H^0(T, \mathcal{O}(n,\lambda)) = 0$. For $n \ge 0$ and $\lambda = 0$, we have*

$$H^0(T, \mathcal{O}_T(n)) \cong \left\{ \nabla_A^{\ (B}\phi^{C\ldots D)} = 0 \right\}, \quad \phi^{C\ldots D} \in \Gamma(\ \mathbb{C}^3, \odot^n \mathcal{O}^A).$$

Proof: A global section of $\mathcal{O}(n,\lambda)$ can be represented as a function $h(\mathcal{U},\mathcal{K})$, where $\mathcal{U} \in \mathbb{C}^2 \backslash \{\binom{0}{0}\}$, $\mathcal{K} \in \mathbb{C}^2$, satisfying

$$h(a\mathcal{U}, a(\mathcal{K} + u\mathcal{U})) = a^n e^{\lambda u} h(\mathcal{U},\mathcal{K}),$$

see Appendix to this chapter for details. It is clear that $\mathcal{O}(n,\lambda)$ does not have a (non-trivial) global section when $n < 0$. It is also easy to check that no homogeneous polynomial $h(\mathcal{U},\mathcal{K})$ can have the property that

$$h(\mathcal{U}, \mathcal{K} + u\mathcal{U}) = e^{\lambda u} h(\mathcal{U},\mathcal{K}), \text{ where } \lambda \neq 0.$$

We thus have the vanishing result.

For $n \ge 0$ and $\lambda = 0$, one has, at E_1 level,

$$\begin{array}{|ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \Gamma(\ \mathbb{C}^3, \mathcal{O}^{\overbrace{(C\ldots D)}^{n}}) & \xrightarrow{\nabla_A^{\ B}} & \Gamma(\ \mathbb{C}^3, \mathcal{O}_A^{\ (BC\ldots D)}) & \longrightarrow & *. \\ \hline \end{array}$$

It converges at E_2, i.e. $E_2^{0,0} = E_\infty^{0,0}$, one thus obtains the result. □

First cohomology:

Proposition 2.10 *For $n \geq 1$ one has*

$$H^1(T, \mathcal{O}(-n-2, \lambda)) \cong \left\{\nabla_A{}^B \phi_{B\ldots E} = -\lambda \phi_{AC\ldots E}\right\}, \quad \phi_{B\ldots E} \in \Gamma(\mathbb{C}^3, \odot^n \mathcal{O}_A).$$

Proof: At E_1 level we have

$$\begin{array}{|ccccc}
\Gamma(\mathbb{C}^3, \mathcal{O}_{\underbrace{(BC\ldots E)}_{n}}) & \xrightarrow{\nabla_A{}^B + \lambda\epsilon_A{}^B} & \Gamma(\mathbb{C}^3, \mathcal{O}_{A(C\ldots E)}) & \longrightarrow & * \\
0 & \longrightarrow & 0 & \longrightarrow & 0. \\
\hline
\end{array}$$

It converges at E_2, $E_2^{0,1} = E_\infty^{0,1}$, $E_1^{1,0} = 0$. Therefore one has the result. □

Proposition 2.11 *For $n \geq -1$, one has*

$$H^1(T, \mathcal{O}(n, \lambda)) \cong \frac{\left\{\nabla^{A(H}\psi_A{}^{BC\ldots E)} = \lambda \psi^{(BC\ldots H)}\right\}}{\left\{\nabla_A{}^{(B}\gamma^{C\ldots E)} + \lambda \epsilon_A{}^{(B}\gamma^{C\ldots E)}\right\}},$$

where $\gamma^{C\ldots E} \in \Gamma(\mathbb{C}^3, \odot^n \mathcal{O}^A)$, $\psi_A{}^{BC\ldots E} \in \Gamma(\mathbb{C}^3, \mathcal{O}_A^{\overbrace{(B\ldots E)}^{n+1}})$ and $\odot^{-1}\mathcal{O}^A$ is taken to be vacuous. In particular, when $n = -1$, the above expression becomes

$$H^1(T, \mathcal{O}(-1, \lambda)) \cong \left\{\nabla^{AB}\psi_A = \lambda \psi^B\right\}.$$

Proof: At E_1 level we have

$$\begin{array}{|ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\Gamma(\mathbb{C}^3, \mathcal{O}^{\overbrace{(C\ldots E)}^{n}}) & \xrightarrow{\nabla_A{}^B + \lambda\epsilon_A{}^B} & \Gamma(\mathbb{C}^3, \mathcal{O}_A^{(BC\ldots E)}) & \xrightarrow{\nabla^{AH} + \lambda\epsilon^{AH}} & \Gamma(\mathbb{C}^3, \mathcal{O}^{(BC\ldots H)}). \\
\hline
\end{array}$$

Again, it converges at E_2. One has $E_2^{1,0} = E_\infty^{1,0}$, $E_1^{0,1} = 0$. □

Proposition 2.12

$$H^1(T, \mathcal{O}(-2, \lambda)) \cong \{\Delta\phi = -2\lambda^2\phi\}.$$

where $\phi \in \Gamma(\mathbb{C}^3, \mathcal{O})$ and $\Delta := \nabla_{AB}\nabla^{AB}$.

Proof: At E_2 level we have

$$\begin{tikzcd}
\Gamma(\mathbb{C}^3,\mathcal{O}) \arrow[drr, "D"] & 0 & 0 \\
0 & 0 & \Gamma(\mathbb{C}^3,\mathcal{O}).
\end{tikzcd}$$

Thus $H^1(T,\mathcal{O}(-2,\lambda)) \cong ker\ D : \Gamma(\mathbb{C}^3,\mathcal{O}) \to \Gamma(\mathbb{C}^3,\mathcal{O})$.

To identify D, consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{O}(-2) \arrow[r, "\pi_B"] \arrow[d, "\wr\|"] & \mathcal{O}_B(-1) \arrow[r, "\pi^B"] \arrow[d, "\nabla_A^{\ B} + \lambda\epsilon_A^{\ B}"] & \mathcal{O} \arrow[r] \arrow[d, "\Box"] & 0 \\
\mu^{-1}\mathcal{O}(-2,\lambda) \arrow[r] & \mathcal{O}(-2) \arrow[r, "\pi_B\nabla_A^{\ B} + \lambda\pi_A"] & \mathcal{O}_A(-1) \arrow[r, "\pi_C\nabla^{AC} + \lambda\pi^A"] & \mathcal{O} \arrow[r] & 0.
\end{tikzcd}$$

The operator $\Box$ can be computed to be $\frac{1}{2}(\Delta + 2\lambda^2)$, where $\Delta = \nabla_{AB}\nabla^{AB}$. Note $\Delta = -2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$.

Each row of the above diagram gives us a spectral sequence, and one gets induced maps between the E_n levels of the spectral sequences for all n. The E_2 level of the top row is

$$\begin{tikzcd}
\Gamma(\mathbb{C}^3,\mathcal{O}) \arrow[drr, "d_2"] & 0 & 0 \\
0 & 0 & \Gamma(\mathbb{C}^3,\mathcal{O}),
\end{tikzcd}$$

and since the top row resolves 0, d_2 must be a multiple of the identity. Thus one has

$$\begin{tikzcd}
\Gamma(\mathbb{C}^3,\mathcal{O}) \arrow[dr, "\cong"] \arrow[dd, "\wr\|"] & \\
& \Gamma(\mathbb{C}^3,\mathcal{O}) \arrow[dd, "\Box"] \\
\Gamma(\mathbb{C}^3,\mathcal{O}) \arrow[dr, "D"] & \\
& \Gamma(\mathbb{C}^3,\mathcal{O})
\end{tikzcd}$$

which identifies D with $\Box = \frac{1}{2}(\Delta + 2\lambda^2)$ (up to a constant). Therefore one obtains the result. See Appendix for an explicit contour integral formula for solutions of $\Delta\phi = -2\lambda^2\phi$. $\Box$

2.3 The $\lambda = 0$ case

When $\lambda = 0$, there are some degeneracy phenomena, cf.[18] for some partial results.

Proposition 2.13 *One has*

$$\begin{aligned} H^1(T, \mathcal{O}_T(n-2)) &\cong H^1(T, \mathcal{O}_T(n)), && \textit{for } n \geq -1, && (2.3)\\ H^1(T, \mathcal{O}_T(-n-2)) &\cong \frac{H^1(T, \mathcal{O}_T(-n))}{\mathbb{C}^{n-1}}, && \textit{for } n \geq 2. && (2.4) \end{aligned}$$

Proof: Consider the fibration

$$\begin{array}{ll} T & , \text{coordinate } (\pi_A, \xi)/\!\sim \\ \big\downarrow \rho & \\ \mathbb{P}_1 & , \text{coordinate } [\pi_A]. \end{array}$$

The holomorphic relative de Rham resolution for ρ is

$$0 \longrightarrow \rho^{-1}\mathcal{O}_{\mathbb{P}_1}(n) \xrightarrow{i} \mathcal{O}_T(n) \xrightarrow{\frac{\partial}{\partial\xi}} \mathcal{O}_T(n-2) \longrightarrow 0. \tag{2.5}$$

It gives rise to a long exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(T, \rho^{-1}\mathcal{O}_{\mathbb{P}_1}(n)) \longrightarrow H^0(T, \mathcal{O}_T(n)) \longrightarrow H^0(T, \mathcal{O}_T(n-2)) \\ \longrightarrow H^1(T, \rho^{-1}\mathcal{O}_{\mathbb{P}_1}(n)) \longrightarrow H^1(T, \mathcal{O}_T(n)) \longrightarrow H^1(T, \mathcal{O}_T(n-2)) \longrightarrow 0, \end{aligned}$$

Then by $H^1(T, \rho^{-1}\mathcal{O}_{\mathbb{P}_1}(n)) \cong H^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(n))$ (by Theorem 1.2, as fibre of ρ is $\mathbb{C}$), we obtain the results. $\Box$

Together with the isomorphisms we had in the last section, (2.3), (2.4) can be rewritten as:

Corollary 2.14 *For $n \geq 1$*

$$\frac{\left\{\nabla^{A(H}\psi_A{}^{\overbrace{BC\ldots S)}^{n}} = 0\right\}}{\left\{\nabla_A{}^{(B}\gamma^{C\ldots S)}\right\}} \cong \begin{cases} \{\Delta\phi = 0\} & n \text{ odd} \quad (\dagger) \\ \{\nabla_A{}^B\phi_B = 0\} & n \text{ even} \quad (\ddagger) \end{cases} \tag{2.6}$$

$$\left\{\nabla_A{}^B\phi_{\underbrace{B\ldots E}_{n}} = 0\right\} \cong \begin{cases} \dfrac{\{\Delta\phi = 0\}}{\mathbb{C}^{n-1}\oplus\mathbb{C}^{n-3}\oplus\ldots\oplus\mathbb{C}} & n \text{ even} \quad (\dagger) \\ \dfrac{\{\nabla_A{}^B\phi_B = 0\}}{\mathbb{C}^{n-1}\oplus\mathbb{C}^{n-3}\oplus\ldots\oplus\mathbb{C}^2} & n \text{ odd} \quad (\ddagger). \end{cases} \tag{2.7}$$

Proposition 2.15 *The isomorphisms (2.6), (2.7) can be realized in terms of the following spinorial operations:*

$$\begin{array}{lll} \psi_A{}^{B\ldots S} & \longmapsto \phi = \nabla_{CD}\ldots\nabla_{RS}\psi_B{}^{BCD\ldots RS} & (\dagger) \text{ of } (2.6) \\ \psi_A{}^{B\ldots S} & \longmapsto \phi_A = \nabla_{(BC}\ldots\nabla_{RS}\psi_{A)}{}^{BC\ldots RS} & (\ddagger) \text{ of } (2.6) \\ \phi & \longmapsto \phi_{BC\ldots DE} = \nabla_{(BC}\ldots\nabla_{DE)}\phi & (\dagger) \text{ of } (2.7) \\ \phi_A & \longmapsto \phi_{BC\ldots DEF} = \nabla_{(BC}\ldots\nabla_{DE}\phi_{F)} & (\ddagger) \text{ of } (2.7). \end{array}$$

Proof: Standard calculations, checking field equations are satisfied and the gauges are the kernels of the operations. □

Appendix: Explicit contour integral formula for $\nabla^2\phi = \lambda^2\phi$

The isomorphisms one obtains from the Penrose transform can often be understood in terms of contour integral formulas, cf. e.g. [27,35]. The following is an example in which data on T are explicitly transformed into solutions of $\nabla^2\phi = \lambda^2\phi$ on $\mathbb{C}^3$ via a contour integral formula, where ∇^2 is the Laplacian.

Remark: Recall that the $\lambda = 0$ case was known to Whittaker [36] in 1902.

Let $f : ESL(2,\mathbb{C}) \longrightarrow \mathbb{C}$, $f(gq) = \rho(q^{-1})f(g)$ be a section of $\mathcal{O}(n,\lambda)$, where we take

$$g = \left(\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, X\right) \in ESL(2,\mathbb{C}),\ q = \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} u & w \\ 0 & -u \end{pmatrix}\right) \in Q.$$

Then we have

$$f\left((a\mathcal{U} \quad b\mathcal{U}+a^{-1}\mathcal{V}), \left(\begin{matrix}\mathcal{U} & \mathcal{V}\end{matrix}\right)\begin{pmatrix} u & w \\ 0 & -u \end{pmatrix}\begin{pmatrix} -\widetilde{\mathcal{V}} \\ \widetilde{\mathcal{U}} \end{pmatrix} + X\right) = a^n e^{\lambda u} f((\mathcal{U} \quad \mathcal{V}), X),$$

where $\mathcal{U} = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$, $\mathcal{V} = \begin{pmatrix}\gamma \\ \delta\end{pmatrix}$, $(\mathcal{U} \ \mathcal{V}) = \begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix} \in SL(2,\mathbb{C})$, $\widetilde{\mathcal{U}} = (-\beta \quad \alpha)$, $\widetilde{\mathcal{V}} = (-\delta \quad \gamma)$. Now observe the following,

$$\left(\left(\begin{matrix}\mathcal{U} & \mathcal{V}\end{matrix}\right)\begin{pmatrix} u & w \\ 0 & -u \end{pmatrix}\begin{pmatrix} -\widetilde{\mathcal{V}} \\ \widetilde{\mathcal{U}} \end{pmatrix} + X\right) a\mathcal{U} = a(u\mathcal{U} + X\mathcal{U}).$$

Therefore, when g is sent to gq, we have the following:

$$\begin{cases} \mathcal{U} \longmapsto a\mathcal{U} \\ X\mathcal{U} \longmapsto a(X\mathcal{U} + u\mathcal{U}). \end{cases}$$

We thus can alternatively represent sections of $\mathcal{O}(n,\lambda)$ by holomorphic functions $h(\mathcal{U},\mathcal{K})$, where $\mathcal{U} \in \mathbb{C}^2\backslash\{\binom{0}{0}\}$, $\mathcal{K} \in \mathbb{C}^2$, satisfying

$$h(a\mathcal{U}, a(\mathcal{K} + \mathcal{U})) = a^n e^{\lambda u} h(\mathcal{U},\mathcal{K}).$$

Now consider the following contour integral

$$\phi(X) = \oint_C h(\mathcal{U}, X\mathcal{U})\, \widetilde{\mathcal{U}} d\mathcal{U} \tag{2.8}$$

with $h(a\mathcal{U}, a(X\mathcal{U} + u\mathcal{U})) = a^{-2}e^{\lambda u}h(\mathcal{U}, X\mathcal{U})$. As $\widetilde{\mathcal{U}}d\mathcal{U}$ has homogeneity 2, while $h(\mathcal{U}, X\mathcal{U})$ has homogeneity -2, the integral makes sense as a contour integral over $\mathbb{CP}_1$, where C is a curve winding around the annular region of the intersection of two sets of an open cover of $\mathbb{CP}_1$. One can then check that $\nabla^2\phi = \lambda^2\phi$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Remark: In terms of spinor index notation, (2.8) can be alternatively written as

$$\phi(X^A{}_B) = \oint_C h(\pi^A, X^A{}_B\pi^B)\ \pi_D d\pi^D.$$

Chapter 3

A Holomorphic Penrose Transform for Hyperbolic 3-Space

In this chapter we apply the holomorphic Penrose transform to a twistor correspondence where the target space X is complex hyperbolic 3-space $\mathbb{H}$. The twistor correspondence we study is well-known cf. [18,19], and the Penrose transform was previously discussed in [18], as a symmetry reduction of the Penrose transform in [8]. We present this correspondence independently of the symmetry reduction, using only the intrinsic geometry.

3.1 Correspondence and set-up

The correspondence

Let $\widetilde{\mathbb{H}}$ be the complex Riemannian manifold

$$\widetilde{\mathbb{H}} = \{(t, x, y, z) \in \mathbb{C}^4 \mid t^2 - x^2 - y^2 - z^2 = 1\}$$

with metric induced from $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ on $\mathbb{C}^4$.

Definition 3.1 *Let Q denote the quadric $t^2 - x^2 - y^2 - z^2 = 0$ in $\mathbb{CP}_3 = \mathbb{P}(\mathbb{C}^4)$. Complex hyperbolic space is the complement $\mathbb{H} = \mathbb{CP}_3 \backslash Q$.*

The space $\widetilde{\mathbb{H}}$ is the universal covering of $\mathbb{H}$, and $\mathbb{H}$ has a natural (complex) Riemannian metric induced from $\widetilde{\mathbb{H}}$.

Let $\mathbb{F}$ denote the space

$$\mathbb{F} = \{(p, x) \mid p \in Q, x \in \mathbb{CP}_3 \backslash Q, x \in T_p Q\}.$$

Then letting μ, ν be the obvious forgetful maps, we have a correspondence of complex manifolds

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ Q = \mathbb{P}_1 \times \mathbb{P}_1 & & \mathbb{H} = \mathbb{CP}_3 \backslash Q. \end{tikzcd}$$

Remark: Geodesics in $\mathbb{H}$ are projective lines in $\mathbb{CP}_3$. A vector v at $x \in \mathbb{H}$ is null if the geodesic it generates is tangent to Q.

For $x \in \mathbb{CP}_3 \backslash Q$, $\mu \circ \nu^{-1}(x)$ by definition consists of all p's in Q such that $x \in T_p Q$, which is just the intersection of Q and the polar plane of x. Thus it is biholomorphic to $\mathbb{CP}_1$.

For $p \in Q$, $\nu \circ \mu^{-1}(p)$ consists of all $x \in \mathbb{CP}_3 \backslash Q$ which lie on $T_p Q$. This is a totally geodesic null plane in $\mathbb{H}$. As $\mu^{-1}(p) \cong \mathbb{P}_2 \backslash (\mathbb{P}_1 \cup \mathbb{P}_1) \cong \mathbb{C} \times \mathbb{C}^*$, where $\mathbb{P}_1 \cup \mathbb{P}_1$ corresponds to a degenerate conic (a pair of lines) $T_p Q \cap Q$, is not contractible, we will apply the Penrose transform only to the following restricted correspondence:

$$\begin{tikzcd} & U' \arrow[dl, "\mu |_{U'}"'] \arrow[dr, "\nu |_{U'}"] & \\ U'' & & U, \end{tikzcd}$$

where U is a Stein open subset of $\mathbb{H}$ such that the fibres of $\mu |_{U'}$ are contractible, where open sets $U' := \nu^{-1}(U) \subset \mathbb{F}$, $U'' := \mu \circ \nu^{-1}(U) \subset Q$ respectively.

The coordinates

We write $\mathbb{C}^4$ as $\mathbb{C}^{AA'} \cong \mathbb{C}^A \otimes \mathbb{C}^{A'}$, tensor product of two 2-dimensional complex vector spaces $\mathbb{C}^A, \mathbb{C}^{A'}$ with fixed volume forms $\epsilon_{AB}, \epsilon_{A'B'}$ respectively. Explicitly, in terms of matrices, we have coordinates on $\mathbb{C}^4$:

$$X^{\mathbf{AA'}} = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \in \mathbb{C}^{\mathbf{AA'}},$$

and $\epsilon_{\mathbf{AB}} = \epsilon_{\mathbf{A'B'}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One has $X^{AA'}X_{AA'} = 2(t^2-x^2-y^2-z^2) = \det(X^{AA'})$.

We then give Q, $\mathbb{F}$ and $\mathbb{H}$ the following coordinates

$$([\pi_A], [\xi^{A'}]) \in Q, \qquad ([X^{AA'}], [\pi_A]) \in \mathbb{F}, \qquad [X^{AA'}] \in \mathbb{H},$$

where [ ] means taking the projective equivalence class. The maps μ, ν are

$$\mu : ([X^{AA'}], [\pi_A]) \longmapsto ([\pi_A], [X^{AA'}\pi_A]),$$
$$\nu : ([X^{AA'}], [\pi_A]) \longmapsto [X^{AA'}].$$

Homogeneous spaces

The spaces Q, $\mathbb{F}$ and $\mathbb{H}$ are all homogeneous spaces of the group $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$: One can think of $X^{AA'}$ as a 2×2 matrix, π_A, $\xi^{A'}$ as column vectors. Then $(g,h) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ act on Q, F and $\mathbb{H}$ by

$$\begin{aligned} (g,h)([\pi],[\xi]) &= (g^{t^{-1}}[\pi], h[\xi]) && \text{on } Q, \\ (g,h)([X],[\pi]) &= (g[X]h^t, g^{t^{-1}}[\pi]) && \text{on } \mathbb{F}, \\ (g,h)[X] &= g[X]h^t && \text{on } \mathbb{H}. \end{aligned}$$

Then one can obtain

$$\begin{aligned} Q &= (SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/(P \times P^t), \\ \mathbb{F} &= (SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/(P \times \mathbb{Z}_2), \\ \mathbb{H} &= (SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))/(SL(2,\mathbb{C}) \times \mathbb{Z}_2), \end{aligned}$$

with $P = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha,\beta \in \mathbb{C},\ \alpha \neq 0 \right\}$, where $P \times \mathbb{Z}_2 \hookrightarrow SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ and $SL(2,\mathbb{C}) \times \mathbb{Z}_2 \hookrightarrow SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ are both given by $(g,\pm) \mapsto (g, \pm g^{t^{-1}})$.

Homogeneous line bundles

Definition 3.2 *The holomorphic line bundle $\mathcal{O}(m \mid n)$, $m, n \in \mathbb{Z}$, on Q is the homogeneous vector bundle associated with the following representation of $P \times P^t$:*

$$\rho\left(\left(\begin{pmatrix}\alpha & \beta \\ 0 & \alpha^{-1}\end{pmatrix}, \begin{pmatrix}\alpha'^{-1} & 0 \\ -\beta' & \alpha'\end{pmatrix}\right)\right) = \alpha^{-m}\alpha'^{-n}.$$

Lemma 3.3 *Sections of $\mathcal{O}(m \mid n)$ can be identified with holomorphic functions of homogeneity m in π_A, n in $\xi^{A'}$.*

Proof: Straightforward. □

Definition 3.4 *Define a line bundle $\mathcal{O}(n, \lambda)$ on Q to be*

$$\mathcal{O}(n, \lambda) := \mathcal{O}(\tfrac{n}{2} + \lambda \mid \tfrac{n}{2} - \lambda),$$

where $\frac{n}{2} + \lambda, \frac{n}{2} - \lambda \in \mathbb{Z}$.

Definition 3.5 *The bundles $\mathcal{O}_{\mathbb{F}}(n)$ and $\mathcal{O}_{\mathbb{F}}(n \mid 1)$, $n \in \mathbb{Z}$, on $\mathbb{F}$ are the bundles associated with the representations of $P \times \mathbb{Z}_2$,*

$$\rho\left(\left(\begin{pmatrix}\alpha & \beta \\ 0 & \alpha^{-1}\end{pmatrix}, \pm\begin{pmatrix}\alpha^{-1} & 0 \\ -\beta & \alpha\end{pmatrix}\right)\right) = \alpha^{-n},$$
$$\rho'\left(\left(\begin{pmatrix}\alpha & \beta \\ 0 & \alpha^{-1}\end{pmatrix}, \pm\begin{pmatrix}\alpha^{-1} & 0 \\ -\beta & \alpha\end{pmatrix}\right)\right) = \pm\alpha^{-n}$$

respectively.

One can check easily that

$$\mu^*\mathcal{O}(m \mid n) \cong \begin{cases} \mathcal{O}_{\mathbb{F}}(m+n) & \text{when } n \text{ is even} \\ \mathcal{O}_{\mathbb{F}}(m+n \mid 1) & \text{when } n \text{ is odd.} \end{cases}$$

Definition 3.6 *The line bundle $\mathcal{O}_{\mathbb{H}}(1)$ on $\mathbb{H}$ is the bundle associated with the representation*

$$\rho(g, \pm g^{t^{-1}}) = \pm 1.$$

The Levi-Civita connection on $\mathbb{H}$

Let $\mathcal{O}_A$, $\mathcal{O}_{A'}$ be the restriction to $\mathbb{H}$ of product bundles over $\mathbb{C}^{AA'}$ with fibres $\mathbb{C}_A$, $\mathbb{C}_{A'}$ respectively. They are naturally isomorphic and can be related by

$$v^A = -X^A_{A'}v^{A'}, \quad v^{A'} = X^{A'}_A v^A.$$

If $\Sigma^{AA'}$ is tangent to $\mathbb{H}$, set

$$\sigma^{AB} = -\Sigma^{AA'}X^B_{A'},$$

then σ^{AB} is symmetric automatically by the condition $\Sigma^{AA'}X_{AA'} = 0$. While if $\Sigma^{AA'}$ is not tangent to $\mathbb{H}$, the projection of $\Sigma^{AA'}$ onto $\mathbb{H}$ is given by $-\Sigma^{A'(A}X^{B)}_{A'}$. The tangent bundle of $\mathbb{H}$ can then be identified with $\mathcal{O}^{(AB)}$.

Now the covariant derivative on $\mathbb{H}$ is given by

$$\nabla_{AB}f = 2X^{B'}_{(A}\partial_{B)B'}f \tag{3.1}$$

$$\nabla_{AB}\mu_C = 2X^{B'}_{(A}\partial_{B)B'}\mu_C - \mu_{(A}\epsilon_{B)C}, \tag{3.2}$$

where $\partial_{AA'} := \frac{\partial}{\partial X^{AA'}}$. This is the spinor version of the usual Levi-Civita connection on hyperbolic space as it is torsion-free and $\nabla_{AB}\epsilon_{CD} = 0$.

The relative de Rham resolution and direct images

Proposition 3.7 *For $\frac{n}{2}+\lambda \in \mathbb{Z}$, $\frac{n}{2}-\lambda \in 2\mathbb{Z}$, the relative de Rham sequence is*

$$0 \longrightarrow \mu^{-1}\mathcal{O}(n,\lambda) \xrightarrow{i} \mathcal{O}(n) \xrightarrow{D_A} \mathcal{O}_A(n+1) \xrightarrow{D^A} \mathcal{O}(n+2) \longrightarrow 0.$$

where:

$$D_A f = (\pi_B\nabla_A{}^B - (\tfrac{n}{2}-\lambda)\pi_A)f$$

$$D^A h_A = (\pi_C\nabla^{AC} - (\tfrac{n+1}{2}-\lambda)\pi^A)h_A.$$

Proof: For $\Omega^1_\mu \cong \mathcal{O}_A(1)$ and $\Omega^2_\mu \cong \mathcal{O}(2)$, use arguments similar to that in the proof of Lemma 2.1.

To identify D_A, represent sections of $\mu^{-1}\mathcal{O}(n,\lambda)$ as functions $h(\pi_A, X^{AA'}\pi_A)$, homogeneous of degree $\frac{n}{2}+\lambda$ in π_A and $\frac{n}{2}-\lambda$ in $X^{AA'}\pi_A$. Then explicit calculation shows

$$D_A f = -2\pi^B X^{B'}_{(A}\partial_{B)B'}f - (\tfrac{n}{2}-\lambda)\pi_A f,$$

which equals $(\pi_B \nabla_A{}^B - (\frac{n}{2} - \lambda)\pi_A)f$. The second map D^A can then be determined by $d_\mu \circ d_\mu = 0$. $\square$

The direct images of $\mathcal{O}(n)$ are given in Proposition 0.29.

3.2 Results

We now discuss the Penrose transform in the following cases: $H^0(U'', \mathcal{O}(n, \lambda))$, $H^1(U'', \mathcal{O}(n, \lambda))$: (i) $n < -2$, (ii) $n > -2$, (iii) $n = -2$, where n is an integer, $\frac{n}{2} + \lambda \in \mathbb{Z}$ and $\frac{n}{2} - \lambda \in 2\mathbb{Z}$.

Lemma 3.8 *If $\frac{n}{2} - \lambda$ is odd, the Penrose transform yields essentially the same results as what follow, except that direct image sheaves are twisted by $\mathcal{O}_{\mathbb{H}}(1)$.*

Zeroth cohomology:

Proposition 3.9 *For $n \geq 0$ and $\frac{-n}{2} \leq \lambda \leq \frac{n}{2}$,*

$$H^0(U'', \mathcal{O}(n, \lambda)) \cong \left\{ \nabla_A{}^{(B}\phi^{C\ldots D)} = -\lambda \epsilon_A{}^{(B}\phi^{C\ldots D)} \right\}, \quad \phi^{C\ldots D} \in \Gamma(U, \odot^n \mathcal{O}^A),$$

and $H^0(U'', \mathcal{O}(n, \lambda)) = 0$ otherwise.

Proof: A global section of $\mathcal{O}(n, \lambda)$ can be thought of as a function $f(\pi_A, \xi^{A'})$, homogeneous of degree $\frac{n}{2} + \lambda$ in π_A and $\frac{n}{2} - \lambda$ in $\xi^{A'}$. For such a section to exist, we need the homogeneities on both copies of $\mathbb{P}_1$ to be greater or equal to zero, i.e. $\frac{n}{2} + \lambda \geq 0$ and $\frac{n}{2} - \lambda \geq 0$. Therefore $H^0(U'', \mathcal{O}(n, \lambda)) = 0$ unless $\frac{-n}{2} \leq \lambda \leq \frac{n}{2}$ (thus $n \geq 0$). In particular, when $n = 0$, λ can only be 0.

For $n \geq 0$, the spectral sequence (1.5) in this case gives

$$\begin{aligned} H^0(U'', \mathcal{O}(n, \lambda)) \cong \mathit{ker}\ \nu_*(\pi_B \nabla_A{}^B - (\tfrac{n}{2} - \lambda)\epsilon_A{}^B \pi_B) : \\ \Gamma(U, \odot^n \mathcal{O}^A) \to \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1} \mathcal{O}^A) \end{aligned}$$

The map $\pi_B \nabla_A{}^B - (\frac{n}{2} - \lambda)\epsilon_A{}^B \pi_B : \mathcal{O}(n) \to \mathcal{O}_A(n+1)$ gives

$$\begin{aligned} \pi_C \ldots \pi_E f^{\overbrace{C \ldots E}^{n}} &\longmapsto (\pi_B \nabla_A{}^B - (\tfrac{n}{2} - \lambda)\epsilon_A{}^B \pi_B)(\pi_C \ldots \pi_E f^{C\ldots E}) \\ &= \pi_B \ldots \pi_E (\nabla_A{}^B + \lambda \epsilon_A{}^B) f^{C\ldots E}. \end{aligned}$$

where $f^{C\ldots E}$ is a local section of $\mathcal{O}^{(C\ldots E)}$, and we have used $\pi_B\nabla_A{}^B(\underbrace{\pi_C\ldots\pi_E}_{n}) = \frac{n}{2}\epsilon_A{}^B\pi_B\ldots\pi_E$. Therefore $\nu_*(\pi_B\nabla_A{}^B - (\frac{n}{2}-\lambda)\epsilon_A{}^B\pi_B)$ here is not $\nabla_A{}^B - (\frac{n}{2}-\lambda)\epsilon_A{}^B$, but $\nabla_A{}^B + \lambda\epsilon_A{}^B$ instead. We thus obtain the result. □

First cohomology:

Proposition 3.10 *For $n = -m-2 \leq -3$,*

$$H^1(U'', \mathcal{O}(-m-2,\lambda)) \cong \left\{\nabla_A{}^B\phi_{B\ldots E} = -\lambda\phi_{AC\ldots E}\right\}, \quad \phi_{B\ldots E} \in \Gamma(U, \odot^m\mathcal{O}_A).$$

Proof: The spectral sequence (1.5) gives us

$$H^1(U'', \mathcal{O}(n,\lambda)) \cong ker\ \nu_*^1(\pi_B\nabla_A{}^B - (\tfrac{n}{2}-\lambda)\epsilon_A{}^B\pi_B) : \\ \Gamma(U, \odot^m\mathcal{O}_A) \to \Gamma(U, \mathcal{O}_A \otimes \odot^{m-1}\mathcal{O}_A).$$

To determine $\nu_*^1(\pi_B\nabla_A{}^B - (\frac{n}{2}-\lambda)\epsilon_A{}^B\pi_B)$, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-m-2) \arrow[r, "\overbrace{\pi_B\ldots\pi_F}^{m+1}"] \arrow[d, "\pi_B\nabla_A{}^B - (\frac{n}{2}-\lambda)\epsilon_A{}^B\pi_B"] & \mathcal{O}_{(B\ldots F)}(-1) \arrow[r, "\pi^F"] \arrow[d, "D_A{}^B"] & \mathcal{O}_{(B\ldots E)} \arrow[r] \arrow[d, "\widetilde{D}_A{}^B"] & 0 \\
0 \arrow[r] & \mathcal{O}_A(-m-1) \arrow[r, "\pi_C\ldots\pi_F"] & \mathcal{O}_{A(C\ldots F)}(-1) \arrow[r, "\pi^F"] & \mathcal{O}_{A(C\ldots E)} \arrow[r] & 0.
\end{tikzcd}$$

For the first square to commute, $D_A{}^B$ is computed to be $\nabla_A{}^B - (\frac{1}{2}-\lambda)\epsilon_A{}^B$, where we have used $\nabla_A{}^B\underbrace{\pi_B\ldots\pi_F}_{m+1} = \frac{m+3}{2}\pi_A\ldots\pi_F$. For the second square to commute, $\widetilde{D}_A{}^B$ then is computed to be $\nabla_A{}^B + \lambda\epsilon_A{}^B$.

Therefore, on $\mathbb{H}$, we have

$$\begin{array}{ccc}
\nu_*^1\mathcal{O}(-m-2) & \cong & \mathcal{O}_{(B\ldots E)} \\
\downarrow \nu_*^1(\pi_B\nabla_A{}^B - (\frac{n}{2}-\lambda)\epsilon_A{}^B\pi_B) & & \downarrow \nabla_A{}^B + \lambda\epsilon_A{}^B \\
\nu_*^1\mathcal{O}_A(-m-1) & \cong & \mathcal{O}_{A(C\ldots E)},
\end{array}$$

which leads us to the claimed result. □

Proposition 3.11 *For $n \geq -1$,*

$$H^1(U'', \mathcal{O}(n,\lambda)) \cong \frac{\left\{\nabla^{A(H}\psi_A{}^{BC\ldots E)} = \lambda\psi^{(BC\ldots H)}\right\}}{\left\{\nabla_A{}^{(B}\gamma^{C\ldots E)} + \lambda\epsilon_A{}^{(B}\gamma^{C\ldots E)}\right\}},$$

where $\gamma^{C\ldots E} \in \Gamma(U, \odot^n \mathcal{O}^A)$, $\psi_A{}^{BC\ldots E} \in \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1}\mathcal{O}^A)$ *and* $\odot^{-1}\mathcal{O}^A$ *is taken to be vacuous. In particular, when* $n = -1$, *the above expression becomes*

$$H^1(U'', \mathcal{O}(-1, \lambda)) \cong \{\nabla^{AB}\psi_A = \lambda\psi^B\}.$$

Proof: The spectral sequence (1.5) yields

$$H^1(U'', \mathcal{O}(n, \lambda)) \cong \frac{\{\text{Potentials}\}}{\{\text{Gauge}\}}.$$

where

$$\begin{aligned}
\{\text{Potentials}\} &\cong ker\ \nu_*(\pi_H\nabla^{AH} - (\tfrac{n+1}{2} - \lambda)\epsilon^{AH}\pi_H) : \\
&\qquad \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1}\mathcal{O}^A) \to \Gamma(U, \odot^{n+2}\mathcal{O}^A) \\
\{\text{Gauge}\} &\cong ker\ \nu_*(\pi_B\nabla_A{}^B - (\tfrac{n}{2} - \lambda)\epsilon_A{}^B\pi_B) : \\
&\qquad \Gamma(U, \odot^n\mathcal{O}^A) \to \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1}\mathcal{O}^A).
\end{aligned}$$

Now for $\{\text{Potentials}\}$, $\nu_*(\pi_H\nabla^{AH} - (\frac{n+1}{2} - \lambda)\epsilon^{AH}\pi_H)$ is determined by the following map

$$\begin{aligned}
\underbrace{\pi_B \ldots \pi_E}_{n+1} g_A{}^{B\ldots E} &\mapsto (\pi_H\nabla^{AH} - (\tfrac{n+1}{2} - \lambda)\epsilon^{AH}\pi_H)\pi_B \ldots \pi_E g_A{}^{B\ldots E} \\
&= \pi_H\pi_B \ldots \pi_E(\nabla^{AH} + \lambda\epsilon^{AH})g_A{}^{B\ldots E},
\end{aligned}$$

where $g_A^{B\ldots E}$ is a local section of $\mathcal{O}_A^{(B\ldots E)}$. For $\{\text{Gauge}\}$, $\nu_*(\pi_B\nabla_A{}^B - (\frac{n}{2} - \lambda)\epsilon_A{}^B\pi_B)$ has already been worked out in the proof of Proposition 3.9. We therefore have the result. □

Proposition 3.12 *There is an isomorphism:*

$$H^1(U'', \mathcal{O}(-2, \lambda)) \cong \{\Delta\phi = -2(\lambda^2 - 1)\phi\},$$

where $\Delta := \nabla^{AB}\nabla_{AB}$ *and* $\phi \in \Gamma(U, \mathcal{O})$.

Proof: Consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}(-2) \arrow[r, "\pi_B"] \arrow[d, "\wr\|"] & \mathcal{O}_B(-1) \arrow[r, "\pi^B"] \arrow[d, "\nabla_A{}^B - (\frac{1}{2} - \lambda)\epsilon_A{}^B"] & \mathcal{O} \arrow[d, "\square"] \\
\mathcal{O}(-2) \arrow[r, "\pi_B\nabla_A{}^B - (-1 - \lambda)\epsilon_A{}^B\pi_B"'] & \mathcal{O}_A(-1) \arrow[r, "\pi_C\nabla^{AC} - (-\frac{1}{2} - \lambda)\epsilon^{AC}\pi_C"'] & \mathcal{O}.
\end{tikzcd}$$

For the second square to commute, $\Box$ can be computed to be $\Box = \frac{1}{2}(\Delta + 2(\lambda^2 - 1))$, where $\Delta = \nabla^{AB}\nabla_{AB}$. Then similar arguments as that in the proof of Proposition 2.12 yield the result. $\Box$

Chapter 4

The Penrose Transform for Einstein-Weyl Spaces

The construction of minitwistor spaces for Euclidean space and hyperbolic space are examples of a general construction due to Hitchin [16]. We will review this construction and construct a Penrose transform in this general situation.

In Section 4.1 the geometry of Einstein-Weyl space (cf. [16,19,26]) is introduced and in Section 4.2 we review Hitchin's correspondence (cf. [16,19]). Section 4.3 gives a spinorial treatment of Einstein-Weyl spaces which provides the notation for the Penrose transform discussed in Section 4.4. In Section 4.5 the Penrose transform for mini-ambitwistor spaces (to be defined) is discussed. In Section 4.6, we discuss the relation between mini-ambitwistor spaces and minitwistor spaces.

4.1 Geometry of EW spaces

Basic definitions and formulas

Definition 4.1 *A Weyl connection on a complex conformal 3-manifold is a torsion free connection D_a which preserves the conformal structure $[g_{ab}]$ in the sense that*

$$D_a g_{bc} = -2\omega_a g_{bc} \tag{4.1}$$

for some 1-form ω_a, for each metric g_{ab} in the conformal class of metrics. A complex 3-manifold $\mathcal{W}$ with such a pair of structure $(D_a, [g_{ab}])$ is called a Weyl space.

Note that ω_a depends on the representative g_{ab}. If we have a rescaling

$$g_{ab} \mapsto \Omega^2 g_{ab},$$

then ω_a will change accordingly to

$$\omega_a \mapsto \omega_a - D_a \ln \Omega. \tag{4.2}$$

Let ∇_a be the Levi-Civita connection for g_{ab}, then one has that, when acting on a vector, D_a and ∇_a are related by the formula

$$D_a V^b = \nabla_a V^b + \delta_a^b \omega_k V^k + V^b \omega_a - V_a \omega^b. \tag{4.3}$$

The corresponding formula for D_a, ∇_a acting on 1-forms is

$$D_a V_b = \nabla_a V_b - \omega_a V_b - \omega_b V_a + \omega_c V^c g_{ab}. \tag{4.4}$$

Definition 4.2 *The curvature tensor ${W_{abc}}^d$ of the Weyl connection is defined by*

$$(D_a D_b - D_b D_a) V^d = {W_{abc}}^d V^c, \tag{4.5}$$

where the Ricci tensor and Ricci scalar are defined by

$$W_{ab} = {W_{adb}}^d\ ; \quad W = g^{ab} W_{ab},$$

respectively

Here we follow [29] in sign convention for the Ricci curvature.

Note ${W_{abc}}^d$, W_{ab} are both conformally invariant with weight zero, while $W \mapsto \Omega^{-2} W$ under the conformal rescaling. $W_{abcd} = W_{[ab]cd}$ is clear from the definition.

Lemma 4.3

$$W_{ab(cd)} = 2 D_{[a} \omega_{b]} g_{cd}. \tag{4.6}$$

Proof: By the torsion free condition, $(D_aD_b - D_bD_a)(V_cV^c) = 0$ for all V_c. Then a straightforward calculation gives (4.6). □

From (4.3) and (4.5) (or equivalently (4.4) and $(D_aD_b - D_bD_a)V_c = -W_{abc}{}^dV_d$), one obtains

Proposition 4.4

$$W_{[ab]} = 3\nabla_{[a}\omega_{b]}, \tag{4.7}$$

$$W_{(ab)} = R_{ab} + \nabla_{(a}\omega_{b)} - \omega_a\omega_b + g_{ab}(\nabla_k\omega^k + \omega_k\omega^k), \tag{4.8}$$

$$W = R + 4\nabla_k\omega^k + 2\omega_k\omega^k, \tag{4.9}$$

where R_{ab} is the Ricci curvature for the metric connection ∇_a and $R := R_a{}^a$.

Proof: Standard calculations (cf. [26] also). □

The Bianchi identities are $W_{[abc]d} = 0$ and $D_{[a}W_{bc]d}{}^e = 0$ as usual. Note $D_{[a}W_{bc]de} \neq 0$ in general, however.

Expressing $W_{abc}{}^d$ in terms of Rho tensor $\widetilde{P}_{ab}$

In conformal geometry one sometimes introduces the so-called *Rho tensor*, P_{ab}, by

$$C_{abcd} = R_{abcd} + P_{ac}g_{bd} - P_{bc}g_{ad} + P_{bd}g_{ac} - P_{ad}g_{bc},$$

where C_{abcd} is the Weyl tensor of the Levi-Civita connection and $P_{ab} = P_{(ab)}$. For a Riemannian 3-manifold, as there is no Weyl tensor, the equation just reduces to

$$R_{abcd} = -2P_{[a|c|}g_{b]d} + 2P_{[a|d|}g_{b]c}, \tag{4.10}$$

where P_{ab} and R_{ab} are related by

$$P_{ab} = -R_{ab} + \frac{1}{4}Rg_{ab}.$$

Under a conformal change of metric $g_{ab} \to \Omega^2 g_{ab}$, P_{ab} changes as follows

$$P_{ab} \longmapsto \widehat{P}_{ab} = P_{ab} - \nabla_a\Gamma_b + \Gamma_a\Gamma_b - \frac{1}{2}\Gamma_k\Gamma^k g_{ab}, \tag{4.11}$$

where $\Gamma_a := \nabla_a \ln\Omega$ (cf. [5]).

Definition 4.5 *The tensor $\widetilde{P}_{ab}$ on $\mathcal{W}$ is defined by*

$$\widetilde{P}_{ab} = -W_{(ab)} + \frac{W}{4} g_{ab} - \nabla_{[a}\omega_{b]}. \tag{4.12}$$

Note that, unlike P_{ab}, $\widetilde{P}_{ab}$ is in general not symmetric in ab. The analogous result to (4.10) is

$$W_{abcd} = -2\widetilde{P}_{[a|c|}g_{b]d} + 2\widetilde{P}_{[a|d|}g_{b]c} + 2\nabla_{[a}\omega_{b]}g_{cd}, \tag{4.13}$$

where the existence of the last term is clearly required by (4.6).

Lemma 4.6 *The tensors $\widetilde{P}_{ab}$ and P_{ab} are related by*

$$\widetilde{P}_{ab} = P_{ab} - \nabla_a\omega_b + \omega_a\omega_b - \frac{1}{2}\omega_k\omega^k g_{ab}. \tag{4.14}$$

The relationship between $W_{abc}{}^d$ and $R_{abc}{}^d$ is thus clearly governed by (4.10), (4.13) and (4.14). Explicitly we have

$$\begin{aligned} W_{abc}{}^d &= 2(R_{[a|c|} + \nabla_{[a}\omega_{|c|} - \omega_{[a}\omega_{|c|})g_{b]}{}^d - 2(R_{[a}{}^d + \nabla_{[a}\omega^d - \omega_{[a}\omega^d)g_{b]c} \\ &\quad +(2\omega_k\omega^k - R)g_{[a|c|}g_{b]}{}^d + 2\nabla_{[a}\omega_{b]}g_c{}^d. \end{aligned}$$

Einstein-Weyl equation

Definition 4.7 *A Weyl space $\mathcal{W}$ is said to be an Einstein-Weyl space if, in addition to the Weyl structure $(D_a, [g_{ab}])$, the Einstein-Weyl equation*

$$W_{(ab)} = \frac{1}{3}W g_{ab} \tag{4.15}$$

holds.

Alternatively, with the help of (4.8) and (4.9), this equation can be expressed in terms of the metric connection and its associated curvature tensors, namely

$$R_{ab} + \nabla_{(a}\omega_{b)} - \omega_a\omega_b = \Lambda g_{ab}, \tag{4.16}$$

where $\Lambda = \frac{1}{3}(R + \nabla_k\omega^k - \omega_k\omega^k)$. Hereafter 'Einstein-Weyl' will sometimes be abbreviated as 'EW'.

4.2 Hitchin's correspondence

Let $\mathcal{T}$ be a complex surface containing a rational curve C with normal bundle $\mathcal{O}(2)$. As $H^1(\mathbb{P}_1, \mathcal{O}(2)) = 0$, by Kodaira's theorem [21](Theorem 0.16), C belongs to a locally complete family of curves $\{C_x, x \in \mathcal{W}\}$ for some complex manifold $\mathcal{W}$, where $T_x\mathcal{W}$ is canonically isomorphic to $H^0(C_x, \mathcal{O}(2)) \cong \mathbb{C}^3$.

Definition 4.8 *A complex surface $\mathcal{T}$ admitting rational curves with normal bundles $\mathcal{O}(2)$ is called a minitwistor space.*

Definition 4.9 *A vector V at $x \in \mathcal{W}$ corresponding to a section of $H^0(C_x, \mathcal{O}(2))$ is null if the two zeros of the section are coincident.*

As the set of such V's is a quadric cone, this gives $\mathcal{W}$ a conformal structure, cf.[16].

Definition 4.10 *Given a vector V at $x \in \mathcal{W}$, the geodesic through x along V is the one-dimensional family of curves in $\mathcal{T}$ which meet C_x at the two zeros of V (should V be null, consider the curves tangent to C_x at the double zero).*

This gives $\mathcal{W}$ a projective structure. See [16] for details.

The conformal structure and projective structure are compatible (cf. Definition 0.25), thus, by Lemma 0.26, there exists a distinguished connection D_a in the projective equivalence class of connections such that

$$D_a g_{bc} = \alpha_a g_{bc}$$

for some 1-form α_a. Therefore, $\mathcal{W}$ has a Weyl structure.

Lemma 4.11 *Given a point $p \in \mathcal{T}$, the set of points in $\mathcal{W}$ corresponding to curves in $\mathcal{T}$ passing through p form a totally geodesic null hypersurface in $\mathcal{W}$.*

Proof: See [16]. □

Proposition 4.12 (Hitchin) *The Weyl structure on $\mathcal{W}$ is in fact an EW structure.*

Proof: Consider the integrability condition for totally geodesic null hypersurfaces, see [16]. □

Conversely, given an EW space, the space of all totally geodesic null hypersurfaces has the structure of a minitwistor space (cf. [16]) and one obtains

Theorem 4.13 (Hitchin) *A solution of the EW equation is equivalent to a complex surface with a family of rational curves of self-intersection number 2.*

Euclidean 3-space and hyperbolic 3-space are examples of EW spaces (by forgetting the scales of the metrics) with minitwistor spaces $T\mathbb{CP}_1$ and $\mathbb{P}_1 \times \mathbb{P}_1$ respectively, and in fact they are the only two EW spaces whose distinguished projective connections are metric.

4.3 Spinorial treatment of EW spaces

In this section we discuss the geometry of EW spaces in terms of spinors, which will simplify the calculations involved in the Penrose transform. As $\mathfrak{co}(3, \mathbb{C}) \cong \mathfrak{gl}(2, \mathbb{C})$, where $\mathfrak{co}(3, \mathbb{C})$ is the Lie algebra of the conformal group $CO(3, \mathbb{C})$, the tangent bundle of a conformal 3-manifold $\mathcal{M}$ is isomorphic to $\odot^2 \mathcal{O}^A$, where $\mathcal{O}^A$ is a rank 2 bundle over $\mathcal{M}$. In particular, on an EW space $\mathcal{W}$, we have $\mathcal{O}^A_x \cong H^0(C_x, \mathcal{O}(1))$, $x \in \mathcal{W}$, and the isomorphism is given by $H^0(C_x, \mathcal{O}(2)) \cong \odot^2 H^0(C_x, \mathcal{O}(1))$.

Now for any $S_{ab} \in \mathcal{O}_{(ab)} = \odot^2 \mathcal{O}_a = \odot^2 \mathcal{O}_{(AA')} \cong \mathcal{O}_{(AA'BB')} \oplus \mathcal{O}$, one can write it as

$$S_{ab} = S_{AA'BB'} = S_{(AA'BB')} + \frac{1}{3} S_{CD}{}^{CD} (\epsilon_{(A|B|}\epsilon_{A')B'}),$$

where $[\epsilon_{AB}]$ corresponds to the conformal metric. Note A' has nothing to do with conjugation, we use A' only to indicate the correspondence between spinor indices (AA') with vector indices a. A metric g_{ab} is related to an $\epsilon_{AB} = \epsilon_{[AB]}$ by

$$g_{ab} = \epsilon_{(A|B|}\epsilon_{A')B'}.$$

Similarly, a 2-form $F_{ab} \in \mathcal{O}_{[ab]} = \Lambda^2\mathcal{O}_{(AA')} \cong \mathcal{O}_{(AB)}$ will be written in terms of spinors as

$$F_{ab} = F_{AA'BB'} = F_{(B|(A}\epsilon_{A')|B')},$$

where $F_{(AA')(BB')} = -F_{(BB')(AA')}$ and $F_{AB} = F_{(AB)} = F_{AA'B}{}^{A'}$.

Definition 4.14 *Let ∇_{AB} be a connection on $\mathcal{O}^A$, the* <u>*torsion tensor*</u> *$T_{ab}{}^c = T_{ABCD}{}^{EF}$, a section of $\mathcal{O}_{[ab]}{}^c$, of ∇_{AB} is defined by*

$$\nabla_{AB}\nabla_{CD}f - \nabla_{CD}\nabla_{AB}f = T_{ABCD}{}^{EF}\nabla_{EF}f,$$

for $f \in \Gamma(\mathcal{W}, \mathcal{O})$.

Definition 4.15 *Define the operator $\Box_{AB}$ by*

$$\nabla_{AB}\nabla_{CD}\nu^E - \nabla_{CD}\nabla_{AB}\nu^E - T_{ABCD}{}^{EF}\nabla_{EF}\nu^E = -2\epsilon_{(A|(C}\Box_{D)|B)}\nu^E.$$

Lemma 4.16 *For any ∇_a, $\mathcal{D}_a$, there exist unique connections ∇_{AB}, D_{AB} on $\mathcal{O}^A$ such that they agree with ∇_a, D_a when acting on $\mathcal{O}^a$, and $\nabla_{AB}\epsilon_{CD} = 0$, $D_{AB}\epsilon_{CD} = -\omega_{AB}\epsilon_{CD}$ respectively.*

Remark: Alternatively, the existence of spinorial version of the EW connection can be proved by means of sheaf cohomology techniques [17].

Proof: Recall that by $\mathfrak{co}(3,\mathbb{C}) \cong \mathfrak{gl}(2,\mathbb{C})$, we can write $g_{ab} = \epsilon_{(A|B|}\epsilon_{A')B'}$. Therefore if one can find a 'torsion-free' connection ∇_{AB} satisfying $\nabla_{AB}\epsilon_{CD} = 0$ (which implies $\nabla_{AB}g_{ab} = 0$), then by the uniqueness of the Levi-Civita connection, ∇_{AB} generates ∇_a. We shall now proceed to find such a ∇_{AB} (the uniqueness of ∇_{AB} will be evident from the construction). The corresponding unique D_{AB} for D_a will be constructed subsequently.

Let ∇_{AB} be an arbitrary connection on $\mathcal{O}^A$ (with torsion in general). The torsion tensor $T_{AB}{}^{CDEF}$ can be written as $\epsilon_{(A}{}^{(C}F_{B)}{}^{D)EF}$ by its symmetry, thus defining $F_{AB}{}^{CD}$, where $F_{AB}{}^{CD} = F_{(AB)}{}^{(CD)}$.

Now any other connection $\widehat{\nabla}_{AB}$ is related to ∇_{AB} by

$$\widehat{\nabla}_{AB}\nu^D = \nabla_{AB}\nu^D + Q_{ABC}{}^D\nu^C,$$

where the tensor $Q_{ABC}{}^{D}$ is the *contorsion tensor* (cf. [2]). It has the symmetry $Q_{ABC}{}^{D} = Q_{(AB)C}{}^{D}$. After some calculation one obtains

$$\widehat{F}_{AC}{}^{EF} = F_{AC}{}^{EF} + Q_{(A}{}^{(E}{}_{C)}{}^{F)} - Q_{(A|B|}{}^{B(E}\epsilon_{C)}{}^{F)}.$$

Therefore, given a ∇_{AB}, we need to search for a $Q_{ABC}{}^{D}$ so that $\widehat{F}_{AC}{}^{EF} = 0$. This can be achieved, for example, by letting $Q_{ABC}{}^{D} = -F_{ACB}{}^{D} + \frac{1}{2}F_{(A|K|}{}^{K}{}_{B)}\epsilon_{C}{}^{D} - \frac{1}{4}F_{KL}{}^{KL}\epsilon_{(A|C|}\epsilon_{B)}{}^{D}$. In other words, one can find a $\widehat{\nabla}_{AB}$ which is torsion free.

However, there is a freedom in choosing such a $Q_{ABC}{}^{D}$, namely for

$$\widetilde{Q}_{ABCD} = Q_{ABCD} + \sigma_{(A|C|}\epsilon_{B)D}, \quad \text{where } \sigma_{AB} = \sigma_{(AB)}, \tag{4.17}$$

we have

$$-\widetilde{Q}_{(A}{}^{(E}{}_{C)}{}^{F)} + \widetilde{Q}_{(A|B|}{}^{B(E}\epsilon_{C)}{}^{F)} = -Q_{(A}{}^{(E}{}_{C)}{}^{F)} + Q_{(A|B|}{}^{B(E}\epsilon_{C)}{}^{F)}, \tag{4.18}$$

i.e. $\widetilde{Q}_{ABCD}$ and Q_{ABCD} have the same torsion. This freedom is exactly the right amount to fix the connection on the line bundle ϵ_{AB} such that $\nabla_{AB}\epsilon_{CD} = 0$. One thus obtains the unique connection which, when applied to vectors, gives the Levi-Civita connection.

The connection D_{AB} is then uniquely determined by

$$D_{AB}\nu^{D} = \nabla_{AB}\nu^{D} + \omega_{(A|C|}\epsilon_{B)}{}^{D}\nu^{C}, \tag{4.19}$$

which, by (4.17) and (4.18), is torsion-free and, when applied to vectors, gives the connection D_a. □

The formula (4.19) is the spinorial version of (4.3). It is not difficult to show that the corresponding D_{AB} and ∇_{AB} actions on $\mathcal{O}_A$ are related by

$$D_{AB}\nu_C = \nabla_{AB}\nu_C - \omega_{C(A}\nu_{B)}, \tag{4.20}$$

which gives rise to (4.4).

Definition 4.17 *Define $X_{ABCD} \in \mathcal{O}_{(AB)CD}$ on $\mathcal{W}$ by*

$$\square_{AB}\nu^{D} = D_{(A}{}^{K}D_{B)K}\nu^{D} = X_{ABC}{}^{D}\nu^{C}. \tag{4.21}$$

That $X_{ABCD} = X_{(AB)CD}$ is clear from definition.

Lemma 4.18 *One has the following identity*

$$X_{ABCD} - X_{ABDC} = (D_{(A}{}^{K}\omega_{B)K})\epsilon_{CD}. \tag{4.22}$$

Proof: Standard calculation (apply $D_{(A}^{K} D_{B)K}$ to $\mu_C \nu^C$, with μ_C and ν^C being arbitrary spinor fields and use the torsion-free condition). □

Lemma 4.19 *One has the following formula*

$$\Box_{AB}\nu_C = D_{(A}{}^{K} D_{B)K}\nu_C = -X_{ABC}{}^{D}\nu_D. \tag{4.23}$$

Proof: Standard calculation (with the help of (4.22)). □

Expressing $W_{abc}{}^{d}$ in terms of spinors we have

$$W_{abc}{}^{d} = W_{AA'BB'CC'}{}^{DD'} = -4\epsilon_{(A|(B} X_{B')|A')(C}{}^{(D}\epsilon_{C')}{}^{D')}.$$

The Bianchi Identity $W_{[abc]}{}^{d} = 0$ becomes $X_{(A|B|}{}^{B}{}_{D)} = 0$.

Definition 4.20 *We decompose X_{ABCD} as*

$$X_{ABCD} = S_{ABCD} + \beta_{AB}\epsilon_{CD} - \frac{2}{3}\epsilon_{C(A}\beta_{B)D} + s\epsilon_{(A|C|}\epsilon_{B)D}, \tag{4.24}$$

where S_{ABCD}, β_{AB} and s are sections of $\mathcal{O}_{(ABCD)}$, $\mathcal{O}_{(AB)}[-1]$ and $\mathcal{O}[-2]$ respectively.

Note: Here $\mathcal{O}[n]$ is the sheaf of holomorphic functions of conformal weight n (cf. Section 0.4, 0.6). We will keep the notation $\mathcal{O}[n]$ even when the bundle is canonical trivial so as to keep track of the behavior of sections of the bundle under a conformal change of metric.

Lemma 4.21 *One has the following identities*

$$\begin{aligned} s &= \frac{-1}{12}W, \\ \beta_{AB} &= \frac{3}{4}D_{(A}^{K}\omega_{B)K}, \\ S_{AA'BB'} &= \frac{1}{2}(W_0)_{AA'BB'}, \end{aligned}$$

where $(W_0)_{AA'BB'} = (W_0)_{ab} = W_{(ab)} - \frac{1}{3}Wg_{ab}$ *is the symmetric trace-free part of the Ricci tensor.*

Proof: Standard calculation. $\square$

Therefore the EW equation just corresponds to $S_{ABCD} = 0$.

Analysis on the correspondence space $\mathcal{F}$

Let $\mathcal{F}$ be the total space of the projective spin bundle $\mathbb{P}(\mathcal{O}_A)$ over $\mathcal{W}$. A point on $\mathcal{F}$ is a pair (Σ, x), where Σ is a totally geodesic null hypersurface of $\mathcal{W}$, and $x \in \Sigma$. Now as $T_x\Sigma$ is composed of vectors of the form $\mu^{(A}\pi^{B)}$, where π_A is a fixed spinor field, up to scaling, with $\pi^A\pi^B$ being normal to Σ, and whose integral curves are null geodesics in Σ, we can write the tautologically defined projective spinor field on $\mathcal{F}$ as $[\pi_A]$.

Definition 4.22 *Define $\mathcal{O}(-1)$ on $\mathcal{F}$ by specifying the fibre at $(\Sigma, x) \in \mathcal{F}$ to be $(\mathcal{O}(-1)_\Sigma)_x$, where $\mathcal{O}(-1)_\Sigma$ is the 1-dimensional subbundle of $\mathcal{O}_A \mid_\Sigma$ spanned by $[\pi_A]$.*

Definition 4.23 *The holomorphic vector bundle $\mathcal{O}^{(A\ldots L)}(p)[q]$ on $\mathcal{F}$ is defined by $\mathcal{O}^{(A\ldots L)}(p)[q] := \mathcal{O}^{(A\ldots L)} \otimes \mathcal{O}(p) \otimes \mathcal{O}[q]$, where $\mathcal{O}^{(A\ldots L)}$ and $\mathcal{O}[q]$ are the pullbacks of $\mathcal{O}^{(A\ldots L)}$ and $\mathcal{O}[q] := \odot^q\mathcal{O}^{[AB]}$ from $\mathcal{W}$ to $\mathcal{F}$, respectively, and $\mathcal{O}(n) := \otimes^n\mathcal{O}(-1)^*$.*

Note the definition of $\mathcal{O}(n)$ on $\mathcal{F}$ agree with that in Definition 0.28.

In order to do calculations explicitly, we shall in general assume a choice of spinor field π_A in $[\pi_A]$ on $\mathcal{F}$, and consider π_A as a section of $\mathcal{O}_A(1)$, π^A as a section of $\mathcal{O}^A(1)[-1]$. Then one has an exact sequence

$$0 \to \mathcal{O}(-1) \xrightarrow{\pi_A} \mathcal{O}_A \xrightarrow{\pi^A} \mathcal{O}(1)[-1] \to 0. \tag{4.25}$$

One also has $T_\mu = \mathcal{O}^A(-1)[1]$, $\Omega^1_\mu = \mathcal{O}_A(1)[-1]$, and $\Omega^2_\mu = \mathcal{O}(2)[-3]$, where $T_\mu := \Omega^{1*}_\mu$, and Ω^p_μ are holomorphic relative p-forms on $\mathcal{F}$ with respect to the natural map $\mu : \mathcal{F} \longrightarrow \mathcal{T}$.

Proposition 4.24 *Let $\mathcal{M}$ be a conformal 3-manifold with a compatible projective structure. For a spinor field ξ_A on $\mathcal{M}$ to integrate to totally geodesic null hypersurfaces with normals $\xi^A\xi^B$, one needs*

$$\xi^B\xi^D D_{AB}\xi_D = 0, \tag{4.26}$$

where D_{AB} is the preferred affine connection. The (global) condition for the existence of a 2 parameter family of totally geodesic null hypersurfaces is the EW condition $S_{ABCD} = 0$.

Proof: For ξ_A on $\mathcal{M}$ to give totally geodesic null hypersurfaces, we need: given any $x \in \mathcal{M}$ and a spinor η^A at x, the following equation always has a solution,

$$\mu^A\xi^B D_{AB}(\mu^{(C}\xi^{D)}) = \phi\mu^{(C}\xi^{D)},$$

such that μ^A equals η^A at x, and ϕ is some function on $\mathcal{M}$. After some calculation, this gives us the condition $\xi^B\xi_D D_{AB}\xi^D = 0$, which can be checked to be equivalent to (4.26).

Now, as the integral curves of $\xi^A\xi^B$ are null geodesics, we can rescale ξ_A so that

$$\xi^A\xi^B D_{AB}\xi_D = 0, \tag{4.27}$$

on each totally geodesic null hypersurface Σ. Note (4.26) is independent of the scaling.

From (4.26) and (4.27), one then obtains, (cf. [29] for arguments of this kind)

$$\xi^B D_{KB}\xi_D = \rho\xi_K\xi_D, \tag{4.28}$$

on each Σ, where ρ is a section of $\mathcal{O}[-1]$.

For the existence of all totally geodesic null hypersurfaces, we need: given a spinor ζ^A at x, there exists a spinor field ξ_A, such that ξ^A equals ζ^A at x and satisfies (4.26). Applying $\xi^D\xi^A D_A^{\ K}$ to (4.28), we obtain

$$\xi^D\xi^A\xi^B D_A^{\ K} D_{KB}\xi_D = 0,$$

which, by the definition of X_{ABCD} and its decomposition, gives us the condition

$$\xi^A \xi^B \xi^C \xi^D S_{ABCD} = 0.$$

As x and ζ^A can be arbitrary, we conclude that $S_{ABCD} = 0$. □

Now, in terms of a spinor field $\pi_A \in [\pi_A]$ on $\mathcal{F}$, the integrability condition enables us to have the following definition.

Definition 4.25 *On $\mathcal{F}$, associated to a spinor field $\pi_A \in [\pi_A]$ satisfying (4.26) and (4.27), understood as equations on $\mathcal{F}$, we define ρ, a section of $\mathcal{O}[-1]$ on $\mathcal{F}$ by*

$$\pi^B D_{KB} \pi_D = \rho \pi_K \pi_D, \tag{4.29}$$

where $\pi^B D_{AB}$ is now a differential operator on $\mathcal{F}$.

Note: Under a rescaling of π_A to $g\pi_A$ preserving (4.27), one has $\pi^B D_{AB} g = \lambda \pi_A g$, where λ is some section of $\mathcal{O}[-1]$, and the ρ in (4.29) changes into $\rho + \lambda$. That is, ρ is *dependent* on the spinor field π_A.

4.4 The Penrose transform

One has the holomorphic twistor correspondence

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathcal{T} & & \mathcal{W}. \end{tikzcd}$$

For $p \in \mathcal{T}$, $\nu(\mu^{-1}(p))$ is a totally geodesic null hypersurface in $\mathcal{W}$, while for $x \in \mathcal{W}$, $\mu(\nu^{-1}(x))$ is a curve in $\mathcal{T}$ with normal bundle $\mathcal{O}(2)$.

Lemma 4.26 *On $\mathcal{F}$, one has a natural connection for $\mathcal{O}(-1)$ and $\mathcal{O}[1]$ induced from the connection $\pi^B D_{AB}$ for $\mathcal{O}_A$. Neither connection is flat unless $\beta_{AB} = 0$, but the induced connection for $\mathcal{O}(-4)[1]$ is flat always.*

Proof: Let f be a section of $\mathcal{O}(-1)$. Then, cf. (4.25), $f\pi_A$ is a section of $\mathcal{O}_A$, and by $\pi^B D_{AB}(f\pi_C) = (\pi^B D_{AB} f + f\rho\pi_A)\pi_C$, one obtains a connection for $\mathcal{O}(-1)$ which is given by

$$f \longmapsto \pi^B D_{AB} f + \rho\pi_A f.$$

The curvature for this connection can be easily computed to be $(\frac{1}{3}\beta_{CD}\pi^C\pi^D)\epsilon_{AB}$.

Similarly, if f is a section of $\mathcal{O}[1]$, then $f\epsilon^{AB}$ is a section of $\mathcal{O}^{[AB]}$ and the connection $\pi^B D_{AB}$ for $\mathcal{O}^{[AB]}$ induces a connection for $O[1]$ which is given by

$$f \longmapsto \pi^B D_{AB} f + \omega_{AB}\pi^B f,$$

the curvature of which can be computed to be $(\frac{-4}{3}\beta_{CD}\pi^C\pi^D)\epsilon_{AB}$.

The induced connection for $\mathcal{O}(-4)[1]$ is therefore flat. □

Definition 4.27 *For $n \in \mathbb{Z}$, define line bundles $\mathcal{O}_{\mathcal{T}}(n)$ on $\mathcal{T}$ by letting the fibre at $p \in \mathcal{T}$ be*

$$\mathcal{O}_{\mathcal{T}}(n)_p = \{\textit{solutions of } \pi^B D_{AB} f - n\rho\pi_A f - \tfrac{n}{4}\pi^B\omega_{AB} f = 0 \textit{ on } \Sigma_p\},$$

where Σ_p is the fibre of $\mu : \mathcal{F} \longrightarrow \mathcal{T}$ at p, and f is a section of $\mathcal{O}(n)[\frac{-n}{4}]|_{\Sigma_p}$.

Remark: Since we are only working locally, we assume the existence of $\mathcal{O}[\frac{1}{4}]$.

Lemma 4.28 *The canonical bundle $\kappa = \wedge^2 T^*\mathcal{T}$ of $\mathcal{T}$ is canonically isomorphic to $\mathcal{O}_{\mathcal{T}}(-4)$.*

Proof: Let f be a section of $\mathcal{O}_{\mathcal{T}}(-4)$, i.e. a section of $\mathcal{O}(-4)[1]$ on $\mathcal{F}$ satisfying

$$\pi^B D_{AB} f + 4\rho\pi_A f + \pi^B\omega_{AB} f = 0. \tag{4.30}$$

One can associate with it a 2-form α on $\mathcal{F}$ given by

$$\begin{aligned}&\alpha(X^{AB}D_{AB} + \sigma_A\frac{\partial}{\partial\pi_A}, \widetilde{X}^{AB}D_{AB} + \widetilde{\sigma}_A\frac{\partial}{\partial\pi_A}) \\ &= (X^{AB}\pi_A\pi_B\widetilde{\sigma}_C\pi^C - \widetilde{X}^{AB}\pi_A\pi_B\sigma_C\pi^C)f,\end{aligned} \tag{4.31}$$

where $X^{AB}D_{AB} + \sigma_A \frac{\partial}{\partial \pi_A}$ and $\widetilde{X}^{AB}D_{AB} + \widetilde{\sigma}_A \frac{\partial}{\partial \pi_A}$ are vector fields on $\mathcal{F}$. It can be checked that α is indeed the pullback of a 2-form on $\mathcal{T}$, i.e. it satisfies

$$\begin{aligned} &\text{(i)} \quad \alpha(V_1, V_2) = 0, && \text{if any } V_i \text{ is vertical to } \mu, \\ &\text{(ii)} \quad d\alpha(V_1, V_2, V_3) = 0 && \text{if any } V_i \text{ is vertical to } \mu. \end{aligned}$$

Conversily, given a 2-form ω on $\mathcal{T}$, we let $\alpha = \mu^*\omega$, then (4.31) associate with it an f satisfying (4.30).

This pairing between sections of $\mathcal{O}_{\mathcal{T}}(-4)$ and 2-forms on $\mathcal{T}$ gives us the canonical isomorphism. □

Lemma 4.29 *On $\mathcal{F}$, one has*

$$\pi^A\pi^B(D_{AB}\rho - \omega_{AB}\rho) = \frac{1}{3}\pi^A\pi^B\beta_{AB}. \tag{4.32}$$

Proof: Apply $\pi^A D_A^{\ K}$ to (4.29). □

Proposition 4.30 *Let $n \in \mathbb{Z}$, and ν be a global section of $\mathcal{O}[-1]$ over $\mathcal{W}$ (thus a global section of $\mathcal{O}[-1]$ on $\mathcal{F}$ independent of π_A). Then for the equation*

$$\pi^B D_{AB} f - (n\rho + \nu)\pi_A f - \frac{n}{4}\pi^B\omega_{AB} = 0 \tag{4.33}$$

to have non-trivial solutions on $\mathcal{F}$, where f is a section of $\mathcal{O}(n)[\frac{-n}{4}]$, one needs

$$D_{AB}\nu - \omega_{AB}\nu = 0. \tag{4.34}$$

Note: Even though ρ is dependent on π_A, (4.33) is nonetheless a well-defined equation on $\mathcal{F}$ (it is invariant under any rescaling of π_A preserving (4.27)).

Proof: Applying $\pi^B D_B^{\ A} - ((n+1)\rho + \nu)\pi^A - (\frac{n}{4}+1)\pi^B\omega_B^{\ A}$ to (4.33), we obtain

$$\pi_A\pi^B(D_B^{\ A}(n\rho + \nu) - \omega_B^{\ A}(n\rho + \nu) - \frac{n}{3}\beta_B^{\ A})f = 0. \tag{4.35}$$

By (4.32), we then have

$$\pi^A\pi^B(D_{AB}\nu - \omega_{AB}\nu) = 0,$$

which then yields the result as the term inside the parentheses is independent of π_A. □

When ω_{AB} is closed, (4.34) becomes $D_{AB}\nu = 0$ at the preferred metric, i.e. ν is a constant at the metric. However, when ω_{AB} is not closed, ν can only be zero.

Remark: The geometry of totally geodesic null hypersurfaces may impose additional conditions on the value of ν, e.g. the $\mathbb{H}$ case in Chapter 3.

Definition 4.31 *When (4.33) has nontrivial solutions, we define a line bundle $\mathcal{O}(n,\nu)$ on $\mathcal{T}$ by letting the fibre at $p \in \mathcal{T}$ be*

$$\mathcal{O}(n,\nu)_p = \{\textit{solutions of } \pi^B D_{AB} f - (n\rho + \nu)\pi_A f - \tfrac{n}{4}\pi^B \omega_{AB} = 0 \textit{ on } \Sigma_p\},$$

where Σ_p is the fibre of $\mu : \mathcal{F} \longrightarrow \mathcal{T}$ at p, and f is a section of $\mathcal{O}(n)[\frac{-n}{4}]|_{\Sigma_p}$.

Remark: 1. In general, one only expects $\mathcal{O}_{\mathcal{T}}(n)$ (i.e. $\mathcal{O}(n,0)$) to exist.

2. Any element of $H^1(\mathcal{T},\mathcal{O})$ corresponds to a degree-0 holomorphic line bundle L on $\mathcal{T}$ which can be coupled to fields, cf. [8]. For simplicity here we assume $\mathcal{T} \leftarrow \mathcal{F} \rightarrow \mathcal{W}$ satisfies conditions 1-4 of Chapter 1. Then from Proposition 4.35,

$$H^1(\mathcal{T},\mathcal{O}_{\mathcal{T}}(n)) \cong \left\{D^{A(H}\psi_A^{\ B)} - \omega^{A(H}\psi_A^{\ B)} = 0\right\} / \left\{D_A^{\ B}\gamma\right\},$$

where $\psi_A^{\ B} \in \Gamma(\mathcal{W},\mathcal{O}_A^{\ B}[-1])$, $\gamma \in \Gamma(\mathcal{W},\mathcal{O})$. Write $\psi_A^{\ B} = \nu\epsilon_A^{\ B} + \phi_A^{\ B}$, where ν is a section of $\mathcal{O}[-1]$ and $\phi_{AB} \in \mathcal{O}_{(AB)}$, then the Penrose transform for $\mathcal{O}_{\mathcal{T}}(n) \otimes L$ can be obtained by replacing the D_{AB} in the Penrose transform for $\mathcal{O}(n,\nu)$ by $\widetilde{D}_{AB} = D_{AB} + \phi_{AB}$.

3. The line bundles $\mathcal{O}(0,\nu)$ form a preferred subspace of $H^1(\mathcal{T},\mathcal{O})$ corresponding to $\phi_A^{\ B} = 0$. The condition $D_{AB}\nu - \omega_{AB}\nu = 0$, which is automatic from the equation for $\psi_A^{\ B}$, then says the preferred subspace is 1-dimensional when $\beta_{AB} = 0$, and is just a point when $\beta_{AB} \neq 0$.

The resolution and direct images

The resolution for $\mathcal{O}(n,\nu)$ is

$$0 \rightarrow \mu^{-1}\mathcal{O}(n,\nu) \rightarrow \mathcal{O}(n)[\tfrac{-n}{4}] \xrightarrow{D_A} \mathcal{O}_A(n+1)[\tfrac{-n}{4}-1] \xrightarrow{E^A} \mathcal{O}(n+2)[\tfrac{-n}{4}-3] \rightarrow 0,$$

where

$$\begin{aligned} D_A &= \pi^B D_{AB} - (n\rho + \nu)\pi_A - \tfrac{n}{4}\pi^B \omega_{AB}, \\ E^A &= \pi^B D_B^{\ A} - ((n+1)\rho + \nu)\pi^A - (\tfrac{n}{4} + 1)\pi^B \omega_B^{\ A}. \end{aligned}$$

We have $E^A D_A = 0$ by (4.35).

For direct images of $\mathcal{O}(n)$, see Proposition 0.29.

The Isomorphisms

Let U be a Stein open subset of $\mathcal{W}$ such that the fibres of $\mu\mid_{U'}: U' := \nu^{-1}(U) \longrightarrow U'' := \mu \circ \nu^{-1}(U) \subset \mathcal{T}$ are contractible. We also assume that Condition 4 of chapter 1 holds for the restricted correspondence $U'' \longleftarrow U' \longrightarrow U$.

We discuss the Penrose transform in the following cases: $H^0(U'', \mathcal{O}(n,\nu))$, $H^1(U'', \mathcal{O}(n,\nu))$: (i) $n = -m-2 < -2$, (ii) $n > -2$, (iii) $n = -2$.

Zeroth cohomology:

Proposition 4.32 *For $n < 0$, $H^0(U'', \mathcal{O}(n,\nu)) = 0$. For $n \geq 0$, one has*

$$H^0(U'', \mathcal{O}(n,\nu)) \cong \left\{ D_A^{\ (B}\phi^{C\ldots D)} - \frac{n}{4}\omega_A^{\ (B}\phi^{C\ldots D)} = -\nu\epsilon_A^{\ (B}\phi^{C\ldots D)} \right\},$$

where $\phi^{C\ldots D} \in \Gamma(U, \odot^n \mathcal{O}^A[\frac{-n}{4}])$.

Proof: The $n < 0$ case is clear. For $n \geq 0$, the spectral sequence (1.5) gives us

$$\begin{aligned} H^0(U'', \mathcal{O}(n,\nu)) &\cong \mathit{ker}\ \nu_*(\pi^B D_{AB} - (n\rho + \nu)\pi_A - \tfrac{n}{4}\pi^B\omega_{AB}) : \\ &\qquad \Gamma(U, \odot^n \mathcal{O}^A[\tfrac{-n}{4}]) \to \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1}\mathcal{O}^A[\tfrac{-n}{4} - 1]). \end{aligned}$$

Now, to know what $\nu_*(\pi^B D_{AB} - (n\rho+\nu)\pi_A - \frac{n}{4}\pi^B\omega_{AB})$ is, we consider the following. On $\mathcal{F}$, the map $(\pi^B D_{AB} - (n\rho+\nu)\pi_A - \frac{n}{4}\pi^B\omega_{AB}) : \mathcal{O}(n)[\frac{-n}{4}] \to \mathcal{O}_A(n+1)[\frac{-n}{4} - 1]$ looks like

$$\begin{aligned} \pi_C \ldots \pi_E f^{\overbrace{C\ldots E}^{n}} &\longmapsto (\pi^B D_{AB} - (n\rho + \nu)\pi_A - \tfrac{n}{4}\pi^B\omega_{AB})(\pi_C \ldots \pi_E f^{C\ldots E}) \\ &= \pi^B \pi_C \ldots \pi_E (D_{AB} + \nu\epsilon_{AB} - \tfrac{n}{4}\omega_{AB}) f^{C\ldots E} \\ &= -\pi_B \pi_C \ldots \pi_E (D_A^{\ B} + \nu\epsilon_A^{\ B} - \tfrac{n}{4}\omega_A^{\ B}) f^{C\ldots E}, \end{aligned}$$

with $f^{C...E}$ being a section of $\mathcal{O}^{(C...E)}[\frac{-n}{4}]$, where we have used $\pi^B D_{AB}(\underbrace{\pi_C \ldots \pi_E}_{n}) = n\rho\pi_A\pi_C \ldots \pi_E$. Therefore, the direct image map is readily identified to be

$$\begin{aligned}&\nu_*(\pi^B D_{AB} - (n\rho+\nu)\pi_A - \tfrac{n}{4}\pi^B\omega_{AB})(\phi^{C...E}) \\ &= -D_A{}^{(B}\phi^{C...E)} - \nu\epsilon_A{}^{(B}\phi^{C...E)} + \tfrac{n}{4}\omega_A{}^{(B}\phi^{C...E)}.\end{aligned}$$

□

First cohomology:

Lemma 4.33 *On $\mathcal{F}$, we have the formula*

$$D_{AB}\pi_D = \pi_A\pi_B\kappa_D + 2\rho\pi_{(A}\epsilon_{B)D}, \tag{4.36}$$

where κ_A is some section of $\mathcal{O}_A(-1)$.

Proof: The equation (4.28) can be rewritten as

$$\pi^B(D_{AB}\pi_D - 2\rho\pi_{(A}\epsilon_{B)D}) = 0,$$

which is equivalent to (again, cf. [29] for arguments of this kind)

$$D_{AB}\pi_D - 2\rho\pi_{(A}\epsilon_{B)D} = \pi_A\pi_B\kappa_D,$$

where κ_A is some section of $\mathcal{O}_A(-1)$. □

Remark: (4.36) is a useful formula in identifying direct image maps. Note that, however, in the final result of each Penrose transform, neither the π_A-dependent spinor κ_A nor the π_A-dependent ρ would ever appear.

Proposition 4.34 *For $n = -m - 2 < -2$, one has*

$$H^1(U'', \mathcal{O}(-m-2, \nu)) \cong \left\{D_A{}^B\phi_{B...E} - (\frac{n}{4}+1)\omega_A{}^B\phi_{B...E} + \nu\phi_{AC...E} = 0\right\},$$

where $\phi_{B...E} \in \Gamma(U, \odot^m\mathcal{O}_A[\frac{-n}{4}-1])$.

Proof: The spectral sequence (1.5) gives us

$$\begin{aligned}H^1(U'', \mathcal{O}(n,\nu)) \cong ker\ \nu^1_*(\pi^B D_{AB} - (n\rho+\nu)\pi_A - \tfrac{n}{4}\pi^B\omega_{AB}) :& \\ \Gamma(U, \odot^m\mathcal{O}_A[\tfrac{-n}{4}-1]) \to \Gamma(U, \mathcal{O}_A \otimes \odot^{m-1}\mathcal{O}_A[\tfrac{-n}{4}-2]).&\end{aligned}$$

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-m-2)[\frac{-n}{4}] \arrow[r, "\overbrace{\pi_B\dots\pi_F}^{m+1}"] \arrow[d, "L_A"] & \mathcal{O}_{(B\dots F)}(-1)[\frac{-n}{4}] \arrow[r, "\pi^F"] \arrow[d, "E_A{}^B"] & \mathcal{O}_{(B\dots E)}[\frac{-n}{4}-1] \arrow[r] \arrow[d, "\widetilde{E}_A{}^B"] & 0 \\
\arrow[r] & \mathcal{O}_A(-m-1)[\frac{-n}{4}-1] \arrow[r, "\pi_C\dots\pi_F"] & \mathcal{O}_{A(C\dots F)}(-1)[\frac{-n}{4}-1] \arrow[r, "\pi^F"] & \mathcal{O}_{A(C\dots E)}[\frac{-n}{4}-2] \arrow[r] & 0,
\end{tikzcd}$$

where $L_A = \pi^B D_{AB} - (n\rho+\nu)\pi_A - \frac{n}{4}\pi^B\omega_{AB}$. Now by commutativity, $E_A{}^B$ is computed to be $-D_A{}^B - (\nu+\rho)\epsilon_A{}^B - \pi_A\kappa^B + \frac{n}{4}\omega_A{}^B$, where we have used $D_A{}^B \underbrace{\pi_B\dots\pi_F}_{m+1} = -(m+3)\rho\pi_A\pi_C\dots\pi_F$. Similarly $\widetilde{E}_A{}^B$ is computed to be $-D_A{}^B - \nu\epsilon_A{}^B + (\frac{n}{4}+1)\omega_A{}^B$. Taking direct images gives the result. □

Proposition 4.35 *For $n \geq -1$, one has*

$$H^1(U'', \mathcal{O}(n,\nu)) \cong \frac{\left\{D^{A(H}\psi_A{}^{BC\dots E)} - (\frac{n}{4}+1)\omega^{A(H}\psi_A{}^{B\dots E)} = \nu\psi^{(H\dots E)}\right\}}{\left\{D_A{}^{(B}\gamma^{C\dots E)} + \nu\epsilon_A{}^{(B}\gamma^{C\dots E)} - \frac{n}{4}\omega_A{}^{(B}\gamma^{C\dots E)}\right\}},$$

where $\psi_A{}^{B\dots E} \in \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1}\mathcal{O}^A[\frac{-n}{4}-1])$, $\gamma^{C\dots E} \in \Gamma(U, \odot^n\mathcal{O}^A[\frac{-n}{4}])$. and $\odot^{-1}\mathcal{O}^A$ is taken to be vacuous. In particular, when $n=-1$, the above expression becomes

$$H^1(U'', \mathcal{O}(-1,\nu)) \cong \left\{D^{AB}\psi_A - \frac{3}{4}\omega^{AB}\psi_A = \nu\psi^B\right\}.$$

Proof: The spectral sequence (1.5) yields

$$H^1(U'', \mathcal{O}(n,\nu)) \cong \frac{\{\text{Potentials}\}}{\{\text{Gauge}\}},$$

where

$$\begin{aligned}
\{\text{Potentials}\} &\cong ker\ \nu_*(\pi^H D_H^A - ((n+1)\rho+\nu)\pi^A - (\tfrac{n}{4}+1)\pi^H\omega_H^A) : \\
&\qquad \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1}\mathcal{O}^A[\tfrac{-n}{4}-1]) \to \Gamma(U, \odot^{n+2}\mathcal{O}^A[\tfrac{-n}{4}-3]); \\
\{\text{Gauge}\} &\cong ker\ \nu_*(\pi^B D_{AB} - (n\rho+\nu)\pi_A - \tfrac{n}{4}\pi^B\omega_{AB}) : \\
&\qquad \Gamma(U, \odot^n\mathcal{O}^A[\tfrac{-n}{4}]) \to \Gamma(U, \mathcal{O}_A \otimes \odot^{n+1}\mathcal{O}^A[\tfrac{-n}{4}-1]).
\end{aligned}$$

As $\nu_*(\pi^B D_{AB} - (n\rho+\nu)\pi_A - \frac{n}{4}\pi^B\omega_{AB})$ has already been worked out in the proof of proposition 4.32, we need only to see what $\nu_*(\pi^H D_H^A - ((n+1)\rho+\nu)\pi^A - (\frac{n}{4}+1)\pi^H\omega_H^A)$ is. By

$$\begin{aligned}
\underbrace{\pi_B\dots\pi_E}_{n+1} g_A{}^{B\dots E} &\mapsto (\pi^H D_H^A - ((n+1)\rho+\nu)\pi^A - (\tfrac{n}{4}+1)\pi^H\omega_H^A)\pi_B\dots\pi_E g_A{}^{B\dots E} \\
&= \pi^H\pi_B\dots\pi_E(D_H^A - \nu\epsilon_H^A - (\tfrac{n}{4}+1)\omega_H^A)g_A{}^{B\dots E} \\
&= -\pi_H\pi_B\dots\pi_E(D^{AH} + \nu\epsilon^{AH} - (\tfrac{n}{4}+1)\omega^{AH})g_A{}^{B\dots E},
\end{aligned}$$

where ${g_A}^{B\ldots E}$ is a local section of $\mathcal{O}_A^{(B\ldots E)}[\frac{-n}{4}-1]$, it is readily identified. $\square$

Proposition 4.36

$$H^1(U'',\mathcal{O}(-2,\nu)) \cong \left\{(\Delta - \frac{1}{2}D\cdot\omega - D_\omega + \frac{1}{4}\omega^2 + 2\nu^2 - 2s)\phi = 0\right\},$$

where $\Delta := D^{AB}D_{AB}$, $D\cdot\omega := D^{AB}(\omega_{AB})$, $D_\omega := \omega^{AB}D_{AB}$, $\omega^2 := \omega_{AB}\omega^{AB}$ *and* $\phi \in \Gamma(U,\mathcal{O}[\frac{-1}{2}])$.

Proof: At E_2 level of the spectral sequence (1.5) we have

$$\begin{tikzcd} \Gamma(\,U,\mathcal{O}[\frac{-1}{2}]) \arrow[drr, "\mathcal{D}"] & 0 & 0 \\ 0 & 0 & \Gamma(\,U,\mathcal{O}[\frac{-5}{2}]). \end{tikzcd}$$

Thus $H^1(U',\mu^{-1}\mathcal{O}(-2,\nu)) \cong$ *ker* $\mathcal{D} : \Gamma(\,U,\mathcal{O}[\frac{-1}{2}]) \to \Gamma(\,U,\mathcal{O}[\frac{-5}{2}])$. The main problem is to identify the differential operator $\mathcal{D}$. Consider the following commutative diagram.

$$\begin{tikzcd} \to \mathcal{O}(-2)[\frac{1}{2}] \arrow[r, "\pi_B"] \arrow[d, "\wr\|"] & \mathcal{O}_B(-1)[\frac{1}{2}] \arrow[r, "\pi^B"] \arrow[d, "F_A^B"] & \mathcal{O}[\frac{-1}{2}] \to \arrow[d, "\square"] \\ \to \mathcal{O}(-2)[\frac{1}{2}] \arrow[r, "L_A"] & \mathcal{O}_A(-1)[\frac{-1}{2}] \arrow[r, "R^A"] & \mathcal{O}[\frac{-5}{2}] \to, \end{tikzcd}$$

where $L_A = \pi^B D_{AB} + (2\rho-\nu)\pi_A + \frac{1}{2}\pi^B\omega_{AB}$ and $R^A = \pi^C D_C^A + (\rho-\nu)\pi^A - \frac{1}{2}\pi^C\omega_C^A$.

If one knows what $\square$ is, then by the following diagram, cf. the proof of Proposition 2.12,

$$\begin{tikzcd} \Gamma(\,U,\mathcal{O}[\frac{-1}{2}]) \arrow[dr, "\cong"] \arrow[d, "\wr\|"] & \\ \Gamma(\,U,\mathcal{O}[\frac{-1}{2}]) \arrow[dr, "\mathcal{D}"] & \Gamma(\,U,\mathcal{O}[\frac{-1}{2}]) \arrow[d, "\square"] \\ & \Gamma(\,U,\mathcal{O}[\frac{-5}{2}]), \end{tikzcd}$$

which says $\mathcal{D} \propto \Box$, one can identify the operator $\mathcal{D}$ (up to an unimportant factor).

Now, for the first square to commute, $F_A{}^B$ is readily computed to be

$$F_A{}^B = -D_A{}^B - (\nu + \rho)\epsilon_A{}^B - \pi_A\kappa^B - \frac{1}{2}\omega_A{}^B.$$

For the second square to commute, letting ϕ_B be a section of $\mathcal{O}_B(-1)[\frac{1}{2}]$, one needs

$$(\pi^C D_C^A + (\rho - \nu)\pi^A - \frac{1}{2}\pi^C\omega_C^A)(-D_A^B - \pi_A\kappa^B - (\nu+\rho)\epsilon_A{}^B - \frac{1}{2}\omega_A{}^B)\phi_B = \Box(\pi^B\phi_B).$$

To compute the left hand side, we need

Lemma 4.37

- $$\Box_{BC}(\pi^B\phi^C) = \tfrac{8}{3}\beta_{BC}\pi^B\phi^C$$
- $$5\rho\pi_C\kappa^C f + 2(D_{AB}\rho - \omega_{AB}\rho + \tfrac{1}{3}\beta_{AB})\pi^A\phi^B + (\pi_C\kappa^C)^2 f - \pi^A\phi^C\omega_{AC}\pi_B\kappa^B + \pi^A\phi^C\pi_B D_{AC}\kappa^B = sf - 4\rho^2 f,$$

where $f := \pi_B\phi^B$, $s = \frac{-1}{12}W$ *and* $\beta_{AB} = \frac{3}{4}D_{(A}^K\omega_{B)K}$.

Proof: Standard computations (for the second identity, use (4.36) to compute the curvature). □

Using the lemma and $D_{AB}\nu - \omega_{AB}\nu = 0$ to compute the LHS, one obtains

$$\Box(-f) = \tfrac{-1}{2}(\Delta - \tfrac{1}{2}D^{AB}\omega_{AB} - \omega^{AB}D_{AB} + \tfrac{1}{4}\omega_{AB}\omega^{AB} + 2\nu^2 - 2s)f,$$

where $\Delta = D^{AB}D_{AB}$ (note $D^{AB}D_{AB} \neq D_{AB}D^{AB}$ in general). □

Remarks: The results of the previous two chapters are just special cases of the results here. One simply fixes the conformal scale by $D_{AB}\epsilon_{CD} = 0$ and forgets about ω_{AB} and conformal weights. Then ν becomes a constant λ, and $D_{AB} = \nabla_{AB}$.

4.5 Mini-ambitwistor spaces

In this section we apply the holomorphic Penrose transform to the correspondence between complex conformal 3-manifolds and mini-ambitwistor spaces (to be defined). The results of this section are not known to exist in the literature yet.

For every complex (geodesically convex) conformal n-manifold $\mathcal{M}$, the space of null geodesics $\mathcal{N}(\mathcal{M})$ is a Hausdorff $2n-3$ dimensional complex manifold [22]. When $n = 4$, the space $\mathcal{N}(\mathcal{M})$ is called an *ambitwistor space.* For the Penrose transform for ambitwistor spaces, see [9]. In the case when $n = 3$, we shall call $\mathcal{N}(\mathcal{M})$ a *mini-ambitwistor space*, and write it as $\mathcal{N}$.

We have the following double fibration:

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathcal{N} & & \mathcal{M}, \end{tikzcd}$$

where a point of the correspondence space $\mathcal{F}$ is a null geodesic of $\mathcal{M}$ together with a point on it. The ν image of a fibre of μ is a null geodesic in $\mathcal{M}$, while the fibre of ν at $x \in \mathcal{M}$ is all the null geodesics through x, which is biholomorphic to $\mathbb{P}_1$.

Given a spinor field π_A on $\mathcal{M}$ generating null geodesics, we will normally rescale π_A such that $\pi^A\pi^B\nabla_{AB}\pi_C = 0$, where ∇_{AB} is the spinor version of a *metric* connection (note locally $\mathcal{O}^a \cong \mathcal{O}^{(AB)}$). As $\nabla_{AB}\epsilon_{CD} = 0$, taking covariant derivative and raising or lowering of indices always commute.

Definition 4.38 *The line bundle $\mathcal{O}_{\mathcal{N}}(n)$ on $\mathcal{N}$ is defined by*

$$\mathcal{O}_{\mathcal{N}}(n)_p = \{\textit{solutions of } \pi^A\pi^B\nabla_{AB}f = 0 \textit{ on } \gamma_p\}, \quad p \in \mathcal{N},$$

where ∇_{AB} is a metric connection lifted to $\mathcal{F}$, γ_p is the fibre of μ at $p \in \mathcal{N}$, f is a section of $\mathcal{O}(n)|_{\gamma_p}$, and $\mathcal{O}(n)$ on $\mathcal{F}$ is defined as in Definition 0.28 (we identify $\mathcal{F}$ with the total space of $\mathbb{P}(\mathcal{O}_A)$ over $\mathcal{M}$).

Note $\pi^A\pi^B\nabla_{AB}f = 0$ is a first order linear differential equation, thus the solution space on γ_p is $\mathbb{C}$.

The relative de Rham resolution is simply

$$0 \to \mu^{-1}\mathcal{O}_{\mathcal{N}}(n) \to \mathcal{O}(n) \xrightarrow{\pi^A\pi^B\nabla_{AB}} \mathcal{O}(n+2)[-2] \to 0. \tag{4.37}$$

We will sometimes write ∇ for $\pi^A\pi^B\nabla_{AB}$.

The long exact sequence of (4.37) gives us the following isomorphisms (note the open sets $U \subset \mathcal{M}$, $U'' \subset \mathcal{N}$ are to be understood in the same sense as in the previous section).

Zeroth cohomology:

Proposition 4.39 *For $n < 0$, $H^0(U'', \mathcal{O}_{\mathcal{N}}(n)) = 0$. For $n \geq 0$,*

$$H^0(U'', \mathcal{O}_{\mathcal{N}}(n)) \cong \left\{\nabla^{(AB}\phi^{C\ldots E)} = 0\right\}, \quad \phi^{C\ldots E} \in \Gamma(U, \odot^n\mathcal{O}^A).$$

Proof: The $n < 0$ case is clear. For $n \geq 0$, we have

$$0 \to H^0(U'', \mathcal{O}_{\mathcal{N}}(n)) \to H^0(U', \mathcal{O}(n)) \stackrel{\nu_*\nabla}{\to} H^0(U', \mathcal{O}(n+2))[-2] \to 0$$

being exact. Hence

$$H^0(U'', \mathcal{O}_{\mathcal{N}}(n)) \cong ker\ \nu_*\nabla : \Gamma(U, \odot^n\mathcal{O}^A) \to \Gamma(U, \odot^{n+2}\mathcal{O}^A[-2]),$$

where $\nabla : \mathcal{O}(n) \to \mathcal{O}(n+2)[-2]$ gives

$$\begin{aligned} \pi_C \ldots \pi_E f^{\overbrace{C\ldots E}^{n}} &\longmapsto (\pi^A\pi^B\nabla_{AB})(\pi_C \ldots \pi_E f^{C\ldots E}) \\ &= \pi_A \ldots \pi_E \nabla^{AB} f^{C\ldots E}, \end{aligned}$$

with $f^{C\ldots E}$ being a section of $\mathcal{O}^{(C\ldots E)}$. □

First cohomology:

Proposition 4.40 *For $m \geq 2$, one has*

$$\begin{aligned} H^1(U'', \mathcal{O}_{\mathcal{N}}(-m-2)) &\cong \left\{\nabla^{AB}\phi_{ABC\ldots E} = 0\right\}, \quad \phi_{A\ldots E} \in \Gamma(U, \odot^m\mathcal{O}_A[-1]). \\ H^1(U'', \mathcal{O}_{\mathcal{N}}(-3)) &\cong \Gamma(U, \mathcal{O}_A[-1]). \end{aligned}$$

Proof: We have

$$0 \to H^1(U'', \mathcal{O}_{\mathcal{N}}(-m-2)) \to H^1(U', \mathcal{O}(-m-2)) \overset{\nu^1_*\nabla}{\to} H^1(U', \mathcal{O}(-m))[-2] \to 0$$

being exact. Hence

$$H^1(U'', \mathcal{O}_{\mathcal{N}}(-m-2) \cong ker\ \nu^1_*\nabla : \Gamma(U, \odot^m \mathcal{O}_A[-1]) \to \Gamma(U, \odot^{m-2}\mathcal{O}_A[-3]).$$

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-m-2) \arrow[r, "\overbrace{\pi_A\pi_B\pi_C\cdots\pi_F}^{m+1}"] \arrow[d, "\pi^A\pi^B\nabla_{AB}"] & \mathcal{O}_{(A\ldots F)}(-1) \arrow[r, "\pi^F"] \arrow[d, "E^{AB}"] & \mathcal{O}_{(A\ldots E)}[-1] \arrow[r] \arrow[d, "\widetilde{E}^{AB}"] & 0 \\
0 \arrow[r] & \mathcal{O}(-m)[-2] \arrow[r, "\pi_C\cdots\pi_F"] & \mathcal{O}_{(C\ldots F)}(-1)[-2] \arrow[r, "\pi^F"] & \mathcal{O}_{(C\ldots E)}[-3] \arrow[r] & 0.
\end{tikzcd}$$

From $\pi^A\pi^B\nabla_{AB}\pi_C = 0$, one has

$$\nabla_{AB}\pi_C = \pi_{(A}\tau_{B)C},$$

where τ_{AB} is some section of $\mathcal{O}_{AB}$ over $\mathcal{F}$, dependent on π_A. Then, for the first square to commute, E^{AB} can be computed to be $\nabla^{AB}+\tau^{AB}$. By the commutativity of the second square one then obtains $\widetilde{E}^{AB} = \nabla^{AB}$. Taking direct images gives the result. □

Note when $n = -4$, $\{\nabla^{AB}\phi_{AB} = 0\}$ is just the space of closed 2–forms.

Proposition 4.41 *For $m \geq 2$, one has*

$$\begin{aligned}
H^1(U'', \mathcal{O}_{\mathcal{N}}(m-2)) &\cong \frac{\Gamma(U, \odot^m \mathcal{O}^A[-2])}{\{\nabla^{(AB}\phi^{C\ldots E)}\}} \quad \phi^{C\ldots E} \in \Gamma(U, \odot^{m-2}\mathcal{O}). \\
H^1(U'', \mathcal{O}_{\mathcal{N}}(-1)) &\cong \Gamma(U, \mathcal{O}^A[-2]).
\end{aligned}$$

Proof: We have

$$\begin{aligned}
0 \to H^0(U'', \mathcal{O}_{\mathcal{N}}(m-2)) \to H^0(U', \mathcal{O}(m-2)) \overset{\nabla}{\to} H^0(U', \mathcal{O}(m))[-2] \\
\to H^1(U'', \mathcal{O}_{\mathcal{N}}(m-2)) \to 0, \quad m \geq 1
\end{aligned}$$

being exact. □

Proposition 4.42 *When $\mathcal{M}$ is either $\mathbb{C}^3$ or $\mathbb{H}$, one has*

$$H^1(U'', \mathcal{O}_{\mathcal{N}}(-2)) \cong \Gamma(U, \mathcal{O}[-2]) \oplus \Gamma(U, \mathcal{O}[-1]).$$

Proof: We have

$$0 \to H^0(U', \mathcal{O}[-2]) \to H^1(U'', \mathcal{O}_{\mathcal{N}}(-2)) \to H^1(U', \mathcal{O}(-2)) \to 0$$

being exact. That is,

$$0 \to \Gamma(U, \mathcal{O}[-2]) \to H^1(U'', \mathcal{O}_{\mathcal{N}}(-2)) \to \Gamma(U, \mathcal{O}[-1]) \to 0 \text{ is exact.}$$

For cases like $\mathbb{C}^3$ and $\mathbb{H}$ which can be seen as homogeneous spaces with semi-simple isotropy groups, the exact sequence splits naturally. □

4.6 Relation between $\mathcal{N}$ and $\mathcal{T}$

Definition 4.43 *Let X be a complex $(2n+1)$-manifold. A (holomorphic) contact structure on X is a line bundle $L \to X$ together with a holomorphic 1-form $\theta \in \Gamma(X, \Omega^1(L))$ such that $\theta \wedge (d\theta)^{\wedge n} \neq 0$ everywhere. A manifold with a contact structure is a contact manifold.*

Note: An important class of contact manifolds comes from projective cotangent bundles: If (x^i) are local coordinates on a manifold M and the cotangent bundle T^*M has coordinates (x^i, y_i) with respect to a local frame of 1-forms dx^i, then there is a canonical 1-form $\sum y_i dx^i$ on T^*M. This then defines a contact structure on $\mathbb{P}(T^*M)$, cf. [16].

Given a contact manifold X, one has a short exact sequence of vector bundles

$$0 \longrightarrow D \longrightarrow TX \longrightarrow L \longrightarrow 0, \tag{4.38}$$

where D is the distribution associated with the contact structure, i.e for $p \in X$, $D_p = \{t \in T_pX \mid \theta(t) = 0\}$.

Definition 4.44 *A submanifold $Y \hookrightarrow X$ is a Legendre submanifold if $dim(Y) = n$ and $TY \subset D$.*

Proposition 4.45 (LeBrun) *Let $\mathcal{M}$, $\mathcal{N}$ be as in section 4.5, then every $\mathcal{N}$ admits a natural contact structure, and for all $x \in \mathcal{M}$, $Q_x := \mu \circ \nu^{-1}(x)$ is a Legendre submanifold.*

For a proof, see [22].

Proposition 4.46 *A conformal 3-manifold $\mathcal{M}$ has a compatible EW structure iff any curve Q_x in $\mathcal{N}$ has a tubular neighborhood U such that U is isomorphic (as a contact manifold) to a neighborhood $\widetilde{U}$ of $\widetilde{Q_x}$, the lifting of $\pi(Q_x)$ in $\mathbb{P}(T^*Z)$, where Z is a complex surface and $\pi : \mathcal{N} \longrightarrow Z$ is a projection which is transverse to the Q_x's.*

Proof: The ($\Longrightarrow$) part is illustrated in [16], where Z is the corresponding minitwistor space.

($\Longleftarrow$): The projection π defines a line subbundle $B \subset T\mathcal{N}$ over $\mathcal{N}$. Now restrict our attention to $U \subset \mathcal{N}$ of a Q_x. By the isomorphism of U and $\widetilde{U}$, and the definition of the canonical contact structure on $\mathbb{P}(T^*Z)$, $B \mid_U$ is a line subbundle of $D \mid_U$, and as the projection is transverse to Q_x, $B \mid_{Q_x}$ is a subbundle of N, the normal bundle of Q_x in U. Then from the short exact sequence

$$0 \longrightarrow D \mid_{Q_x} \longrightarrow T\mathcal{N} \mid_{Q_x} \longrightarrow L \mid_{Q_x} \longrightarrow 0,$$

we deduce (noting $TQ_x \subset D \mid_{Q_x}$ also) the short exact sequence

$$0 \longrightarrow B \mid_{Q_x} \longrightarrow N \longrightarrow L \mid_{Q_x} \longrightarrow 0.$$

Then, as $L^2 \mid_{Q_x} \cong \wedge^3(T^*\mathcal{N})^* \mid_{Q_x} \cong \mathcal{O}(4)$, cf.[23], we have $N/(B \mid_{Q_x}) \cong \mathcal{O}(2)$. Therefore $\pi(Q_x)$ in Z has normal bundle $\mathcal{O}(2)$. This gives Z a minitwistor structure, and $\mathcal{M}$ thus acquires an EW structure. □

Definition 4.47 *A <u>Legendre foliation</u> $\mathcal{L}$ of a contact manifold is a foliation by Legendre submanifolds.*

A projective cotangent bundle $\mathbb{P}(T^*Z)$ clearly has a canonical Legendre foliation where leaves are fibres of $\mathbb{P}(T^*Z) \longrightarrow Z$.

Definition 4.48 *Two Legendre foliations $\mathcal{L}_1$ of X_1, $\mathcal{L}_2$ of X_2 are* equivalent *if there is a biholomorphic mapping $\psi : X_1 \longrightarrow X_2$ satisfying*

$$(i)\ \psi^* L_2^* = L_1^*, \quad and \quad (ii)\ \psi^* \mathcal{L}_2 = \mathcal{L}_1.$$

where L_i is the line bundle L defined in Definition 4.43, $\psi^* \mathcal{L}_2$ is the foliation of X_1 whose leaves are inverse images under ψ of leaves of $\mathcal{L}_2$.

Theorem 4.49 *Let X be a complex contact manifold with a Legendre foliation $\mathcal{L}$, and Z be the space of leaves, then $\mathcal{L}$ is locally equivalent to the canonical Legendre foliation on $\mathbb{P}(T^*Z)$.*

Proof: This is a complex holomorphic version of a theorem due to Pang, see [25] for details. Note our contact manifolds are referred to as 'contact manifolds in the wider sense' in [25]. □

As an immediate consequence of Lemma 4.46 and Theorem 4.49, we have

Corollary 4.50 *The existence of an EW structure (locally) on $\mathcal{M}$ is equivalent to the existence of a transverse Legendre foliation of $\mathcal{N}$.*

Remark: The question of whether a conformal 3-manifold admits an EW metric locally has been the subject of some attention recently (by Eastwood, Pedersen, Tod, etc — private communication). We see that this question can be translated into a question in the existence of a Legendre foliation, which may be more tractable. We have not so far made any further progress with this approach.

Chapter 5

The Non-holomorphic Penrose Transform

The non-holomorphic Penrose transform was originally introduced by Bailey, Eastwood and Singer [3] as a mechanism which relates sheaf cohomologies on a complex manifold to solutions of systems of PDE's on another *smooth* manifold. It has been successful in unifying various examples of transforms of this nature under a general framework, cf. the Introduction chapter. This method is best presented in terms of *involutive structures*, cf. [34]. However, no treatment of this general Penrose transform (cf. [3]) has appeared in the literature yet. In this chapter, basing on ideas of [3], we introduce the non-holomorphic Penrose transform in terms of involutive structures. Note that the differential operator $\nabla + \Phi$ discussed in an earlier version of [3] is only partially correct.

In Section 5.1 we review some basic material about involutive structures, and then in Section 5.2 we introduce the non-holomorphic Penrose transform. Section 5.3 is a discussion of involutive structures for homogeneous spaces. In Section 5.4, we consider the non-holomorphic Penrose transform for a correspondence where all spaces involved are homogeneous spaces for a symmetry group. Appendix 1 contains proofs of some propositions, and Appendix 2 is a comparison of the non-holomorphic Penrose transform with the realization of a holomorphic Penrose transform.

5.1 Involutive structure

Involutive structure on a manifold

Definition 5.1 *Let M be a smooth manifold, and TM be the complexified tangent bundle. An involutive structure on M is a complex subbundle $T^{0,1} \subset TM$ such that*

$$[T^{0,1}, T^{0,1}] \subset T^{0,1}. \tag{5.1}$$

An involutive manifold is a smooth manifold together with an involutive structure.

Lemma 5.2 *An involutive structure $T^{0,1} \subset TM$ is a complex structure iff*

$$TM = T^{0,1} \oplus \overline{T^{0,1}}.$$

Definition 5.3 *Let $\mathcal{E}^1$ be the dual of TM. Define the vector bundle $\mathcal{E}^{1,0}$ by*

$$\mathcal{E}^{1,0}_x := \{\omega \in \mathcal{E}^1_x \mid \langle \omega, v \rangle = 0, \forall v \in T^{0,1}_x\}, \quad x \in M,$$

where $\langle \omega, v \rangle$ is the usual pairing between a vector space and its dual. The bundle $\mathcal{E}^{0,1}$ is defined by the exactness of

$$0 \to \mathcal{E}^{1,0} \to \mathcal{E}^1 \to \mathcal{E}^{0,1} \to 0. \tag{5.2}$$

Lemma 5.4 *The condition (5.1) is equivalent to*

$$d\mathcal{E}^{1,0} \subset \mathcal{E}^{1,0} \wedge \mathcal{E}^1, \tag{5.3}$$

i.e. $\mathcal{E}^{1,0}$ generates a differential ideal in $\mathcal{E}^\bullet$.

Proof: See, e.g. [34]. □

Therefore we can alternatively use $\mathcal{E}^{1,0}$ satisfying (5.3) to denote an involutive structure.

Definition 5.5 *The bundles $\mathcal{E}^{0,p}$ and $\mathcal{E}^{p,0}$, $p \in \mathbb{N}$, are defined by*

$$\mathcal{E}^{0,p} := \wedge^p \mathcal{E}^{0,1}, \qquad \mathcal{E}^{p,0} := \wedge^p \mathcal{E}^{1,0}.$$

Because of (5.3), one can obtain a quotient complex

$$(\Gamma(M, \mathcal{E}^{0,\bullet}), \bar{\partial})$$

from $(\Gamma(M, \mathcal{E}^{\bullet}), d)$, where $\Gamma(M, \mathcal{E}^{0,\bullet})$ is identified with $\Gamma(M, \mathcal{E}^{\bullet})/\langle\Gamma(M, \mathcal{E}^{1,0})\rangle$, and the differential $\bar{\partial}$ is the natural operator induced from the exterior derivative d.

Definition 5.6 *The involutive cohomology $H^p(M)$ of an involutive manifold M is the cohomology $H^p(\Gamma(M, \mathcal{E}^{0,\bullet}), \bar{\partial})$.*

Involutive structure and compatible vector bundles

Lemma 5.7 *If E is a smooth complex vector bundle over M, and there is a linear map $\bar{\partial} : \Gamma(M, E) \to \Gamma(M, E \otimes \mathcal{E}^{0,1})$ satisfying*

$$\bar{\partial}(fs) = f\bar{\partial}s + (\bar{\partial}f)s, \quad f \in \mathcal{E}(M), s \in \Gamma(M, E),$$

then there is an unique extension of $\bar{\partial}$ to $\bar{\partial} : \Gamma(M, \mathcal{E}^{0,q} \otimes E) \to \Gamma(M, \mathcal{E}^{0,q+1} \otimes E)$ satisfying

$$\bar{\partial}(fs) = (\bar{\partial}f)s + (-1)^q f \wedge \bar{\partial}s, \quad f \in \mathcal{E}^{0,q}(M), s \in \Gamma(M, E).$$

Proof: Standard results. □

Definition 5.8 *A complex vector bundle $E \to M$ is compatible with the involutive structure if there exists a*

$$\bar{\partial} : \Gamma(M, E) \to \Gamma(M, E \otimes \mathcal{E}^{0,1})$$

such that

(i) $\bar{\partial}(fs) = f\bar{\partial}s + (\bar{\partial}f)s, \quad f \in \mathcal{E}(M), s \in \Gamma(M, E),$

(ii) $\bar{\partial}^2 = 0.$

Note $\bar{\partial}^2$ in **(ii)** makes sense because of Lemma 5.7.

Definition 5.9 *A section $s \in \Gamma(M, E)$ is said to be compatible with the involutive structure of M (or simply an 'involutive section') if $\bar{\partial}s = 0$.*

Definition 5.10 *The involutive cohomology $H^q(M, E)$ of E is the cohomology*

$$H^q(\Gamma(M, E \otimes \mathcal{E}^{0,\bullet}), \bar{\partial}).$$

One important example of compatible vector bundles is $\mathcal{E}^{p,0}$, where $\bar{\partial}$ is induced from d. It gives rise to

$$H^q(M, \mathcal{E}^{p,0}) = H^{p,q}(M) = H^q(\Gamma(M, \mathcal{E}^{p,0} \otimes \mathcal{E}^{0,\bullet})).$$

Relations between involutive structures

Definition 5.11 *A map $f : M \to N$ between two involutive manifolds is said to be compatible with the involutive structures if*

$$df(T^{0,1}M) \subset T^{0,1}N. \tag{5.4}$$

Lemma 5.12 *The condition (5.4) is equivalent to*

$$f^*\mathcal{E}_N^{1,0} \subset \mathcal{E}_M^{1,0}. \tag{5.5}$$

Lemma 5.13 *Let $f : M \to N$ be a map compatible with the involutive structures of M and N. If $f^*\mathcal{E}_N^{1,0} \to \mathcal{E}_M^1$ has constant rank, and E is a vector bundle on N compatible with the involutive structure of N, then f^*E is compatible with the involutive structure of M.*

Proof: Using $d\mathcal{E}_N^{1,0} \subset \mathcal{E}_N^{1,0} \wedge \mathcal{E}_N^1$ and the fact that $f^*\mathcal{E}_N^{1,0} \to \mathcal{E}_M^1$ has constant rank, it is easy to see that (5.3) holds for $f^*\mathcal{E}_N^{1,0}$ on M also. That is, $f^*\mathcal{E}_N^{1,0}$ is an involutive structure on M. Hence there exists a complex $(\Gamma(M, \wedge^\bullet(\mathcal{E}_M^1/f^*\mathcal{E}_N^{1,0})), \bar{\partial})$. Furthermore, by the compatibility of E with $\mathcal{E}_N^{1,0}$, we have

$$\bar{\partial} : \Gamma(M, f^*E) \to \Gamma(M, f^*E \otimes \mathcal{E}_M^1/f^*\mathcal{E}_N^{1,0}),$$

satisfying the conditions of Definition 5.8. That is, f^*E is compatible with the involutive structure $f^*\mathcal{E}_N^{1,0}$ on M.

Now, by the compatibility of f, one has $f^*\mathcal{E}_N^{1,0} \subset \mathcal{E}_M^{1,0}$ (thus $\mathcal{E}_M^1/f^*\mathcal{E}_N^{1,0} \twoheadrightarrow \mathcal{E}_M^{0,1}$, cf. Lemma 5.15 to come). Therefore f^*E is also compatible with $\mathcal{E}_M^{1,0}$. □

Definition 5.14 *Let M be a manifold equipped with two involutive structures $T^{0,1}$, $T'^{0,1}$, where $T^{0,1} \subset T'^{0,1} \subset TM$, then one defines the bundle $\mathcal{E}^1_r$ by the exactness of the following sequence*

$$0 \to \mathcal{E}'^{1,0} \to \mathcal{E}^{1,0} \to \mathcal{E}^1_r \to 0. \tag{5.6}$$

Lemma 5.15 *The exact sequence (5.6) is equivalent to the following short exact sequence:*

$$0 \to \mathcal{E}^1_r \to \mathcal{E}'^{0,1} \to \mathcal{E}^{0,1} \to 0. \tag{5.7}$$

Theorem 5.16 *Let E be a vector bundle compatible with the primed involutive structure, then one has the following spectral sequence*

$$E_1^{p,q} = H^q(M, E \otimes \mathcal{E}^p_r) \Rightarrow H'^{p+q}(M, E),$$

where the prime $'$ in $H'^{p+q}(M, E)$ indicates that the cohomology is taken with respect to the primed involutive structure.

Proof: Tensoring (5.7) with E, we obtain an exact sequence

$$0 \to \mathcal{E}^1_r(E) \to \mathcal{E}'^{0,1}(E) \to \mathcal{E}^{0,1}(E) \to 0, \tag{5.8}$$

where $\mathcal{E}^1_r(E)$, $\mathcal{E}'^{0,1}(E)$ and $\mathcal{E}^{0,1}(E)$ denote the resulting vector bundles $\mathcal{E}^1_r \otimes E$ etc. Note the compatibility of E with the primed involutive structure implies the compatibility with the unprimed, and $\mathcal{E}^1_r$ is also compatible with the unprimed.

Consider the filtration of $\Gamma(M, \mathcal{E}'^{0,\bullet}(E))$ induced by (5.8). From the spectral sequence of a filtered complex we obtain

$$E_1^{p,q} = H^q(\Gamma(M, \mathcal{E}^{0,\bullet} \otimes \mathcal{E}^p_r(E))) \Rightarrow H^{p+q}(\Gamma(M, \mathcal{E}'^{0,\bullet}(E))).$$

However, by Definition 5.10, this is nothing but

$$E_1^{p,q} = H^q(M, E \otimes \mathcal{E}^p_r) \Rightarrow H'^{p+q}(M, E). \qquad \square$$

5.2 Non-holomorphic Penrose transform

The non-holomorphic Penrose transform translates holomorphic data on a complex manifold Z to data on a smooth manifold X, where Z and X are related in the following way

Condition 1. There exists a *correspondence of smooth manifolds*, cf. Definition 0.18,

$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ Z & & X. \end{tikzcd} \tag{5.9}$$

Condition 2. Z is a complex manifold, and $\eta(\tau^{-1}(x))$ is a compact complex submanifold of Z, $\forall x \in X$.

Condition 3. Fibres of η are contractible.

An extra **Condition 4** will be introduced when the 'push-down mechanism' is discussed.

Definition 5.17 **(1)** *The involutive structure $\mathcal{E}_Z^{1,0}$ on Z is the complex structure on Z.* **(2)** *The involutive structure on X is given by $\mathcal{E}_X^{0,1} = 0$.* **(3)** *The involutive structure on F is defined by $\mathcal{E}_F^{0,1} = \mathcal{E}_\tau^{0,1}$, where $\mathcal{E}_\tau^1 := \mathcal{E}_F^1/\tau^*\mathcal{E}_X^1$ and, by Condition 2, $\mathcal{E}_\tau^1 = \mathcal{E}_\tau^{1,0} \oplus \mathcal{E}_\tau^{0,1}$.*

Lemma 5.18 *The maps η and τ are compatible with these involutive structures.*

The pull-back mechanism

Proposition 5.19 *Let E be a holomorphic vector bundle on Z, then one has*

$$H_A^r(F, \eta^* E) \cong H^r(Z, \mathcal{O}(E)), \tag{5.10}$$

where $H^r_A(F,V) := H^r(\Gamma(F, V \otimes \mathcal{E}^{0,\bullet}_A))$, *and* $\mathcal{E}^{1,0}_A := \eta^*\mathcal{E}^{1,0}_Z$.

Proof: We can compute the cohomology of $\mathcal{E}^{1,0}_A$ in terms of that of $\mathcal{E}^{1,0}_\alpha := \eta^*\mathcal{E}^1_Z$, where (5.7) becomes

$$0 \to \eta^*\mathcal{E}^{0,1}_Z \to \mathcal{E}^{0,1}_A \to \mathcal{E}^1_\eta \to 0, \tag{5.11}$$

where $\mathcal{E}^1_\eta := \mathcal{E}^1_F/\eta^*\mathcal{E}^1_Z$.

The vector bundle η^*E is compatible with the involutive structure of F, cf. Lemma 5.13. Then by Theorem 5.16, one has

$$E_1^{p,q} = H^q(\Gamma(F, \eta^*E \otimes \eta^*\mathcal{E}^{0,p}_Z \otimes \mathcal{E}^\bullet_\eta)) \Longrightarrow H^r(\Gamma(F, \eta^*E \otimes \mathcal{E}^{0,\bullet}_A)). \tag{5.12}$$

The thing on the left of (5.12) computes the fibre de Rham cohomology with values in $\eta^*E \otimes \eta^*\mathcal{E}^{0,p}_Z$. As the fibres of η are contractible, we have

$$H^q(F, \mathcal{E}^\bullet_\eta(\eta^*E \otimes \eta^*\mathcal{E}^{0,p}_Z)) \cong \begin{cases} \Gamma(Z, E \otimes \mathcal{E}^{0,p}_Z) & q = 0 \\ 0 & q \neq 0. \end{cases}$$

Therefore the spectral sequence just computes the Dolbeault cohomology of the base and gives us $H^p(Z, \mathcal{O}(E))$.

On the other hand, the right hand side of (5.12) is by definition $H^r_A(F, \eta^*E)$. Therefore one obtains the isomorphism. □

Spectral sequence argument

Proposition 5.20 *Let E be as above, one has the spectral sequence*

$$E_1^{p,q} = H^q(\Gamma(F, \eta^*E \otimes \mathcal{E}^{p,0}_\eta \otimes \mathcal{E}^{0,\bullet}_\tau)) \Longrightarrow H^r_A(F, \eta^*E), \tag{5.13}$$

where $\mathcal{E}^{1,0}_\eta := \mathcal{E}^{1,0}_F/\eta^*\mathcal{E}^{1,0}_Z$.

Proof: Use the cohomology of $\mathcal{E}^{1,0}_F$ to compute the cohomology of $\mathcal{E}^{1,0}_A = \eta^*\mathcal{E}^{1,0}_Z$. The sequence (5.7) becomes

$$0 \to \mathcal{E}^{1,0}_\eta \to \mathcal{E}^{0,1}_A \to \mathcal{E}^{0,1}_\tau \to 0. \tag{5.14}$$

Again applying Theorem 5.16 immediately gives the result. □

The push-down mechanism

Definition 5.21 *Let V be a vector bundle on F compatible with the involutive structure $\mathcal{E}_F^{1,0}$. We use $Œ(V)$ to denote the sheaf of germs of involutive sections of V.*

Proposition 5.22 *One has the isomorphism*

$$H^q(\Gamma(F, \eta^* E \otimes \mathcal{E}_\eta^{p,0} \otimes \mathcal{E}_\tau^{0,\bullet})) \cong \Gamma(X, \tau_*^q Œ(\eta^* E \otimes \mathcal{E}_\eta^{p,0})). \tag{5.15}$$

Proof: The results follows as $H^q(\Gamma(F, \eta^* E \otimes \mathcal{E}_\eta^{p,0} \otimes \mathcal{E}_\tau^{0,\bullet}))$ computes the cohomology of the fibre Dolbeult resolution of $Œ(\eta^* E \otimes \mathcal{E}_\eta^{p,0})$. □

To avoid possible dimension jumps of $H^q(\tau^{-1}(x), Œ(\eta^* E \otimes \mathcal{E}_\eta^{p,0}))$ at exceptional points $x \in X$, we shall assume

Condition 4: The rank of $H^q(\tau^{-1}(x), Œ(\eta^* E \otimes \mathcal{E}_\eta^{p,0}))$ is constant as $x \in X$ varies.

Since τ has compact fibres and condition 4 is assumed, $\tau_*^q Œ(\eta^* E \otimes \mathcal{E}_\eta^{p,0})$ is a sheaf of germs of sections of a smooth vector bundle over X.

Summary

Theorem 5.23 *Given a correspondence (5.9) satisfying conditions 1-4 of this chapter, and a holomorphic vector bundle $E \longrightarrow Z$, there is a spectral sequence*

$$E_1^{p,q} = \Gamma(X, \tau_*^q Œ(\eta^* E \otimes \mathcal{E}_\eta^{p,0})) \Longrightarrow H^r(Z, \mathcal{O}(E)). \tag{5.16}$$

We refer to this as the non-holomorphic Penrose transform.

5.3 Involutive structure on homogeneous spaces

Proposition 5.24 *There is a 1-1 correspondence between G-invariant involutive structures on G/H and complex Lie algebras $\mathfrak{y}$ satisfying*

$$\mathfrak{h} \subset \mathfrak{y} \subset \mathfrak{g},$$

where $\mathfrak{g}$ is the complexified Lie algebra of G as described in Section 0.3. A homogeneous space G/H together with an involutive structure determined by $\mathfrak{y}$ will be denoted $(G/H, \mathfrak{y})$.

Proof: Recall $T(G/H)$ is associated to the H-module $\mathfrak{g}/\mathfrak{h}$. Here one has

$$\begin{array}{ccc} T^{0,1} & \subset & TM \\ \updownarrow & & \updownarrow \\ \mathfrak{y}/\mathfrak{h} & \subset & \mathfrak{g}/\mathfrak{h}, \end{array}$$

where $\updownarrow$ indicates the correspondence between a homogeneous vector bundle on G/H and an H-module, cf. Chapter 0. □

Proposition 5.25 *The short exact sequence of vector bundles (5.2) is associated to the short exact sequence of H-modules:*

$$0 \to (\mathfrak{g}/\mathfrak{y})^* \to (\mathfrak{g}/\mathfrak{h})^* \to (\mathfrak{y}/\mathfrak{h})^* \to 0. \tag{5.17}$$

Proposition 5.26 *The involutive manifold $(G/H, \mathfrak{y})$ is a complex manifold if and only if*

$$\mathfrak{g} = \mathfrak{y} + \bar{\mathfrak{y}} \text{ and } \mathfrak{h} = \mathfrak{y} \cap \bar{\mathfrak{y}}.$$

Proof: As $\mathfrak{g}$ is a complexified Lie algebra, there exists a complex conjugation operation $\bar{\ }$ on it with $\mathfrak{g} = \bar{\mathfrak{g}}$ and $\mathfrak{h} = \bar{\mathfrak{h}}$. Then Lemma 5.2 says (cf. [20] also):

$$\mathfrak{g}/\mathfrak{h} = \bar{\mathfrak{y}}/\mathfrak{h} \oplus \mathfrak{y}/\mathfrak{h},$$

which then gives the result. □

Definition 5.27 *A $(\mathfrak{y}, H)$-module is a representation ρ of H and $\dot{\rho}$ of $\mathfrak{y}$ on a vector space such that $\dot{\rho}\mid_{\mathfrak{h}}$ is the derivative of ρ, and such that*

$$\rho(h)\dot{\rho}(X)\rho(h)^{-1} = \dot{\rho}(Ad(h)X) \quad h \in H, X \in \mathfrak{y}. \tag{5.18}$$

Note: (5.18) is automatic when H is connected. For simplicity, we shall hereafter assume that H is connected.

Lemma 5.28 *Homogeneous (finite dimensional) vector bundles $V = G \times_H V_0$ on $(G/H, \mathfrak{y})$, compatible with the involutive structure, are in one-one correspondence with $(\mathfrak{y}, H)$-modules V_0. Involutive sections of V are characterized by*

$$Xs + \dot{\rho}(X)s = 0 \quad X \in \mathfrak{y}, \tag{5.19}$$

where $Xs : G \to V_0$ is defined as the result of applying the left invariant vector field X to s.

Proof: Essentially mimic the proof of Theorem 3.6 of [33] where the case of homogeneous holomorphic vector bundle is considered. □

Note: Here X is used both to stand for an element of $\mathfrak{y}$ and the associated left invariant vector field, and Xs can be understood explicitly as

$$(Xs)(g) = \frac{d}{dt} s(g \exp(tX)) \mid_{t=0},$$

where X is a complex vector field and we treat the real and imaginary parts separately.

Proposition 5.29 *Let $(G/H_1, \mathfrak{y}_1)$, $(G/H_2, \mathfrak{y}_2)$ be two involutive manifolds, and $H_1 \subset H_2$. Then the natural map $f : G/H_1 \longrightarrow G/H_2$ is compatible with the involutive structures if and only if $\mathfrak{y}_1 \subset \mathfrak{y}_2$.*

Involutive cohomologies

Definition 5.30 *Let $(G/H, \mathfrak{y})$ be an involutive manifold and V_0 a $(\mathfrak{y}, H)$-module. Denote the involutive cohomology $H^p(\Gamma(G/H, \mathcal{E}^{0,\bullet}(V)), \bar{\partial})$ by $H^p_{\mathfrak{y}}(G/H, V_0)$.*

Lemma 5.31 *A section of $\mathcal{E}^{0,p} \otimes V$ can be characterized as a smooth function $\alpha : G \to \wedge^p \mathfrak{y}^* \otimes V_0$ satisfying*

- $\alpha(X_1, \dots, X_p) = 0$ *if any* $X_i \in \mathfrak{h}$, (5.20)
- $(Z + \dot{\rho}(Z))\alpha(X_1, \dots, X_p) - \sum_{i=1}^{p} \alpha(X_1, \dots, [Z, X_i], \dots, X_p) = 0, \quad \forall Z \in \mathfrak{h}$.(5.21)

The map

$$\bar{\partial} : \Gamma(G/H, \mathcal{E}^{0,p} \otimes V) \longrightarrow \Gamma(G/H, \mathcal{E}^{0,p+1} \otimes V)$$

is characterized by

$$\begin{aligned} \bar{\partial}\alpha(X_0, \dots X_p) &= \sum_{i=0}^{p} (-1)^i (X_i + \dot{\rho}(X_i))\alpha(X_0, \dots \hat{X}_i, \dots, X_p) \\ &+ \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p), \end{aligned} \quad (5.22)$$

where all $X_i \in \mathfrak{y}$.

Proof: See Appendix 1. □

Proposition 5.32 *Let $(G/H, \mathfrak{y})$, $(G/H, \mathfrak{y}')$ be two involutive structures on G/H such that $\mathfrak{y} \subset \mathfrak{y}'$, then (5.7) is associated to the short exact sequence of $(\mathfrak{y}, H)$-modules*

$$0 \to (\mathfrak{y}'/\mathfrak{y})^* \to (\mathfrak{y}'/\mathfrak{h})^* \to (\mathfrak{y}/\mathfrak{h})^* \to 0.$$

Proposition 5.33 *Elements of $\Gamma(\mathcal{E}^{0,q} \otimes \mathcal{E}^p_r \otimes V)$ can be thought of as smooth functions*

$$\alpha : G \longrightarrow \wedge^q \mathfrak{y}^* \otimes \wedge^p \mathfrak{y}'^* \otimes V_0$$

satisfying

- $\alpha(X_1, \dots, X_q, Y_1, \dots, Y_p) = 0$ *if any* $X_i \in \mathfrak{h}$ *or* $Y_i \in \mathfrak{y}$,
- $(Z + \dot{\rho}(Z))\alpha(X_1, \dots, X_q, Y_1, \dots, Y_p) - \sum_{i=1}^{q} \alpha(X_1, .., [Z, X_i], .., X_q, Y_1, .., Y_p)$ $- \sum_{i=1}^{p} \alpha(X_1, \dots, X_q, Y_1, \dots, [Z, Y_i], \dots, Y_p) = 0, \quad \forall Z \in \mathfrak{h}$.

The map $\bar{\partial}$ is given by

$$\bar{\partial}\alpha(X_0, \dots, X_q, Y_1, \dots, Y_p) =$$

$$\sum_{i=0}^{q}(-1)^i(X_i+\dot\rho(X_i))\alpha(X_0,\ldots,\hat X_i,\ldots,X_q,Y_1,\ldots,Y_p)$$
$$+\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\hat X_i,\ldots,\hat X_j,\ldots,X_q,Y_1,\ldots,Y_p)$$
$$+\sum_{i,j}(-1)^{i+j}\alpha(X_0,\ldots,\hat X_i,\ldots,X_q,[X_i,Y_j],Y_1,\ldots,\hat Y_j,\ldots,Y_p), \quad (5.23)$$

where all $X_i \in \mathfrak{h}$ and all $Y_i \in \mathfrak{h}'$.

Proof: This is obtained simply by using the formula for

$$\bar\partial : \Gamma(\mathcal{E}^{0,q}\otimes E) \to \Gamma(\mathcal{E}^{0,q+1}\otimes E),$$

as is given in (5.22), where E is associated to the $(\mathfrak{h}, H)$-module $\wedge^p(\mathfrak{h}'/\mathfrak{h})^*\otimes V_0$, with the representation of $\mathfrak{h}$ on $\wedge^p(\mathfrak{h}'/\mathfrak{h})^*$ induced by the co-adjoint representation. □

The E_1 differential of $E_1^{p,q} = H^q(\Gamma(G/H,\mathcal{E}^{0,\bullet}\otimes\mathcal{E}^p_\tau\otimes V))$, which we shall call ∂, can also be characterized explicitly. To do so, we consider an arbitrary splitting $\mathfrak{h}' = \mathfrak{h}\oplus\mathfrak{s}$ and write $X = X'\oplus X''$, where $X'\in\mathfrak{h}$, and $X''\in\mathfrak{s}$.

Proposition 5.34 *Sections of $E_1^{p,q} = H^q(\Gamma(G/H,\mathcal{E}^{0,\bullet}\otimes\mathcal{E}^p_\tau\otimes V))$ can be represented as smooth functions*

$$\alpha : G \longrightarrow \wedge^q\mathfrak{h}^*\otimes\wedge^p\mathfrak{s}^*\otimes V_0$$

satisfying $\bar\partial\alpha = 0$, cf. Proposition 5.33 (replacing $Y_i\in\mathfrak{h}'$ with their components in $\mathfrak{s}$), modulo $\bar\partial\gamma$, where $\gamma\in\Gamma(G/H,\mathcal{E}^{0,q-1}\otimes\mathcal{E}^p_\tau\otimes V)$. The map ∂ is then characterized by

$$\partial\alpha(X_1,\ldots,X_q,Y_0,\ldots,Y_p) =$$
$$\sum_{i=0}^{p}(-1)^{i+q}(Y_i+\dot\rho(Y_i))\alpha(X_1,\ldots,X_q,Y_0,\ldots,\hat Y_i,\ldots,Y_p)$$
$$+\sum_{i<j}(-1)^{i+j+q}\alpha(X_1,\ldots,X_q,[Y_i,Y_j]'',Y_0,\ldots,\hat Y_i,\ldots,\hat Y_j,\ldots,Y_p)$$
$$+\sum_{i,j}(-1)^{i+j+q}\alpha([Y_i,X_j]',X_1,\ldots,\hat X_j,\ldots,Y_0,\ldots,\hat Y_i,\ldots,Y_p), \quad (5.24)$$

where all $X_i\in\mathfrak{h}$ and all $Y_i\in\mathfrak{s}$.

Proof: The first part of the proposition is clear from the definition of a cohomology group.

To obtain the formula for ∂, we 'apply' Lemma 5.31 to the involutive structure $(G/H, \mathfrak{y}')$ with the vector bundle taken to be associated to $V_0 \otimes \wedge^q(\mathfrak{y}/\mathfrak{h})^*$ (see Note 1.), and restrict $\wedge^p(\mathfrak{y}'/\mathfrak{h})^*$ to $\wedge^p(\mathfrak{y}'/\mathfrak{y})^*$ at the end.

Note: 1. As $\wedge^q(\mathfrak{y}/\mathfrak{h})^*$ is in fact *not* a $(\mathfrak{y}', H)$-module, there is no justification of applying Lemma 5.31 to this case directly. Nevertheless, as ∂ is a map between cohomology groups, it can be checked that the fact $\bar{\partial}\alpha = 0$ and $\partial\alpha \sim \partial\alpha + \bar{\partial}\gamma$ guarantees that the map is independent of the splitting $\mathfrak{y}' = \mathfrak{y} \oplus \mathfrak{s}$. Therefore this formula does make sense to characterize ∂.

2. As ∂ operates on cohomology classes, we need to check that (i) $\bar{\partial}\alpha = 0$ implies $\bar{\partial}\partial\alpha = 0$, (ii) $\partial\bar{\partial}\gamma \in \mathrm{Im}\bar{\partial}$. As our formulae for $\bar{\partial}$ and ∂ both come from using the formula for $\bar{\partial}'$, with suitable restrictions at the end, it is not difficult to check that these requirements are satisfied. □

5.4 Non-holomorphic Penrose transform in the homogeneous case

Consider a correspondence which satisfies all the four conditions listed in section 5.2, and is of the following form

$$\begin{tikzcd} & (G/M, \mathfrak{r}) \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ (G/L, \mathfrak{q}) & & (G/K, \mathfrak{k}), \end{tikzcd}$$

where G is a real Lie group with subgroups K, L, and $M = L \cap K$, $\mathfrak{r} = \mathfrak{k} \cap \mathfrak{q}$. As the fibres of τ are complex submanifolds of G/M, one has

Proposition 5.35

$$\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{l}, \quad \mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{g},$$
$$\mathfrak{r} \cap \bar{\mathfrak{r}} = \mathfrak{m}, \quad \mathfrak{r} + \bar{\mathfrak{r}} = \mathfrak{k}.$$

Proof: Use the fact that $(G/L, \mathfrak{q})$ and $(K/M, \mathfrak{r})$ are complex manifolds. □

We will in general assume that $\mathfrak{k}$ admits a K-invariant complement $\mathfrak{p}$ in $\mathfrak{g}$, such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Note that $\mathfrak{p}$ is *not* a Lie algebra.

Lemma 5.36 *The two short exact sequences of vector bundles (5.11), (5.14) are associated to*

$$0 \to (\mathfrak{q}/\mathfrak{l})^* \to (\mathfrak{q}/\mathfrak{m})^* \to (\mathfrak{l}/\mathfrak{m})^* \to 0, \tag{5.25}$$
$$0 \to (\mathfrak{q}/\mathfrak{r})^* \to (\mathfrak{q}/\mathfrak{m})^* \to (\mathfrak{r}/\mathfrak{m})^* \to 0 \tag{5.26}$$

respectively.

Theorem 5.37 *There is a spectral sequence*

$$E_1^{p,q} = \Gamma(G/K, \tau^q_{*\mathfrak{r}}(E_0 \otimes \wedge^p(\mathfrak{q}/\mathfrak{r})^*) \Longrightarrow H^{p+q}_{\mathfrak{q}}(G/L, E_0), \tag{5.27}$$

where $\tau^q_{*\mathfrak{r}}$ indicates that the fibre cohomology it is associated with is the involutive cohomology of $\mathfrak{r}$. For a $(\mathfrak{r}, M)$-module V_0, $\tau^q_{*\mathfrak{r}}(V_0)$ is just $\tau^q_* \mathbb{E}(V)$.

Differential operators on G/K

Definition 5.38 *For every $X \in \mathfrak{q}$, define $X^{\mathfrak{p}}$ and $X^{\mathfrak{k}}$ by the unique decomposition*

$$X = X^{\mathfrak{p}} + X^{\mathfrak{k}},$$

where $X^{\mathfrak{p}} \in \mathfrak{p}$, $X^{\mathfrak{k}} \in \mathfrak{k}$.

Definition 5.39 *Let V be a homogeneous vector bundle on $(G/M, \mathfrak{r})$ compatible with the involutive structure. Define two maps D, Φ,*

$$D,\ \Phi\ : \Gamma(G/M, \mathbb{E}(\mathcal{E}_\eta^{p,0} \otimes V)) \longrightarrow \Gamma(G/M, \mathbb{E}(\mathcal{E}_\eta^{p+1,0} \otimes V)),$$

by:

$$D\alpha(Y_0, \ldots Y_p) = \sum_{i=0}^{p}(-1)^i Y_i^{\mathfrak{p}} \alpha(Y_0, \ldots \hat{Y}_i, \ldots, Y_p), \tag{5.28}$$

$$\Phi\alpha(Y_0, \ldots, Y_p) = \sum_{i=0}^{p}(-1)^i (Y_i^{\mathfrak{k}} + \dot{\rho}(Y_i))\alpha(Y_0, \ldots \hat{Y}_i, \ldots, Y_p)$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_p), \tag{5.29}$$

where all $Y_i \in \mathfrak{q}$ and $\alpha : G \longrightarrow \Lambda^p \mathfrak{q}^* \otimes V_0$ satisfies

- $\alpha(Y_1, \ldots, Y_p) = 0$ *if any $Y_i \in \mathfrak{r}$,*
- $(Z + \dot{\rho}(Z))\alpha(Y_1, \ldots, Y_p) - \sum_{i=1}^{p} \alpha(Y_1, \ldots, [Z, Y_i], \ldots, Y_p) = 0, \quad \forall Z \in \mathfrak{r}.$

Their corresponding q^{th} direct image maps $\tau_*^q D$, $\tau_*^q \Phi$ will be denoted $\widetilde{D}$ and $\widetilde{\Phi}$ respectively, and $\widetilde{\Phi}$ will later be shown to be algebraic (Proposition 5.43).

Proposition 5.40 *If E is a homogeneous vector bundle on G/L compatible with the involutive structure on G/L, $V = \eta^* E$, then*

$$\tau_*^q \partial : \tau_*^q \mathbb{E}(\mathcal{E}_\eta^{p,0}(V)) \to \tau_*^q \mathbb{E}(\mathcal{E}_\eta^{p+1,0}(V))$$

is equal to $\widetilde{D} + \widetilde{\Phi}$.

Proof: From Proposition 5.34, it is clear that $\partial = D + \Phi$, (consider $q = 0$). $\square$

Lemma 5.41 *Every homogeneous vector bundle V on G/K inherits an invariant connection ∇ from the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

Proof: Let s be a section of V, then $\nabla s \in \Gamma(G/K, \mathcal{E}_X^1 \otimes V)$ can be characterized by

$$\langle X, \nabla s \rangle = Xs, \quad X \in \mathfrak{p}, \tag{5.30}$$

where we have identified $\mathcal{E}_X^1$ with the homogeneous vector bundle associated to the K-module $\mathfrak{p}^*$. $\square$

Proposition 5.42 *The map $\widetilde{D}$ is the composition of the following maps*

$$\tau_*^q \mathbb{E}(\mathcal{E}_\eta^{p,0}(V)) \xrightarrow{\nabla} \mathcal{E}_X^1 \otimes \tau_*^q \mathbb{E}(\mathcal{E}_\eta^{p,0}(V)) \xrightarrow{\tau_*^q \pi} \tau_*^q \mathbb{E}(\mathcal{E}_\eta^{1,0} \otimes \mathcal{E}_\eta^{p,0}(V))$$
$$\xrightarrow{\tau_*^q \wedge} \tau_*^q \mathbb{E}(\mathcal{E}_\eta^{p+1,0}(V)), \tag{5.31}$$

where ∇ is the invariant connection on G/K, while π is characterized by

$$\pi\beta(X; Y_1, \ldots, Y_p) = \beta(X^{\mathfrak{p}}; Y_1, \ldots, Y_p), \quad X \in \mathfrak{q} \text{ and all } Y_i \in \mathfrak{q},$$

and $\wedge$ is characterized by

$$\wedge\gamma(Y_0, Y_1, \ldots, Y_p) = \sum_{i=0}^{p} (-1)^i \gamma(Y_i; Y_0, \ldots, \hat{Y}_i, \ldots, Y_p), \text{ where all } Y_i \in \mathfrak{q}.$$

Note we have used $\mathcal{E}_X^1 \otimes \tau_*^q \mathbb{E}(\mathcal{E}_\eta^{p,0}(V)) \cong \tau_*^q \mathbb{E}(\mathcal{E}_X^1 \otimes \mathcal{E}_\eta^{p,0}(V))$.

Proof: See Appendix 1. □

Proposition 5.43 *The map $\widetilde{\Phi}$ is algebraic and is characterized by*

$$\begin{aligned}
&\widetilde{\Phi}_x \mu(X_1, \ldots, X_q, Y_0, \ldots, Y_p) = \\
&\quad \sum_{i=0}^{p} (-1)^{i+q} (Y_i^{\mathfrak{k}} + \dot{\rho}(Y_i)) \mu(X_1, \ldots, X_q, Y_0, \ldots, \hat{Y}_i, \ldots, Y_p) \\
&\quad + \sum_{i<j} (-1)^{i+j+q} \mu(X_1, \ldots, X_q, [Y_i, Y_j]'', Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_p) \\
&\quad + \sum_{i,j} (-1)^{i+j+q} \mu([Y_i, X_j]', X_1, \ldots, \hat{X}_j, \ldots, Y_0, \ldots, \hat{Y}_i, \ldots, Y_p),
\end{aligned} \tag{5.32}$$

for all $x \in X$, where all $X_i \in \mathfrak{r}$ and all $Y_i \in \mathfrak{s}$, $\mu \in H^q_{\mathfrak{r}}(K/M, \wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0)$ is represented by $\mu : K \longrightarrow \wedge^q \mathfrak{r}^* \otimes \wedge^p \mathfrak{s}^* \otimes V_0$, $\mathfrak{q} = \mathfrak{r} \oplus \mathfrak{s}$, satisfying $\bar{\partial}_{\mathfrak{r}} \mu = 0$, where $\bar{\partial}_{\mathfrak{r}}$ is the $\bar{\partial}$ associated with $(G/M, \mathfrak{r})$ which we shall sometimes write as $\bar{\partial}_\tau$.

Proof: The stalk of the direct image sheaf $\tau_*^q \mathbb{E}(\mathcal{E}_\eta^{p,0}(V))$ at $x \in X$ is associated to the K-module $H^q_{\mathfrak{r}}(K/M, \wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0)$, noting that $\tau^{-1}(x) \cong K/M$. Then by (5.24) (letting $\mathfrak{y}' = \mathfrak{q}$ and $\mathfrak{y} = \mathfrak{r}$) and Definition 5.39, $\widetilde{\Phi}$ can be seen to be induced by $\widetilde{\Phi}_x$, a map between K-modules, at every $x \in X$, which is characterized in (5.32). Thus $\widetilde{\Phi}$ is algebraic. Note that however $\widetilde{D}$ can *not* be described in terms of "$\widetilde{D}_x$" (note $Y^{\mathfrak{p}}\mu$ does not make sense). □

Note: If $\mathfrak{q} = \mathfrak{r} \oplus (\mathfrak{p} \cap \mathfrak{q})$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, then the formula above reduces to

$$\begin{aligned}&\widetilde{\Phi}_x\mu(X_1, \ldots, X_q, Y_0, \ldots, Y_p)\\ &= \sum_{i=0}^{p}(-1)^{i+q}\dot{\rho}(Y_i)\mu(X_1, \ldots, X_q, Y_0, \ldots, \hat{Y}_i, \ldots, Y_p).\end{aligned}$$

Appendix 1: Proofs of some propositions

Lemma 5.44 *If ξ is a p-form on a manifold M, then $d\xi$ is given by the formula*

$$\begin{aligned}d\xi(X_0, \ldots X_p) &= \sum_{i=0}^{p}(-1)^i L_{X_i}\xi(X_0, \ldots \hat{X}_i, \ldots, X_p)\\ &+ \sum_{0 \le i < j \le p}(-1)^{i+j}\xi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p), \qquad (5.33)\end{aligned}$$

where $L_{X_i}\xi$ is the Lie derivative of ξ with respect to X_i.

For a proof, see e.g. [11].

Proof of Lemma 5.31: The condition (5.20) is by definition required.

Recall the formula for representing a section of a homogeneous vector bundle, cf. Chapter 0. Here we take the vector bundle to be the one associated to $\wedge^p(\mathfrak{y}/\mathfrak{h})^* \otimes V_0$, and obtain

$$\alpha(gh)(X_1, \ldots, X_p) = \rho^{-1}(h)\alpha(g)(\mathrm{Ad}_h X_1, \ldots, \mathrm{Ad}_h X_p),$$

where $g \in G$, $h \in H$ and all $X_i \in \mathfrak{y}$. Note we have used

$$(\mathrm{Ad}^*(h^{-1})\alpha)(\omega) = \alpha(\mathrm{Ad}(h)\omega), \quad \omega \in \wedge^p\mathfrak{y}.$$

Considering its differential form, we have

$$\begin{aligned}&(Z\alpha)(g)(X_1, \ldots, X_p) = \frac{d}{dt}\alpha(ge^{tZ})(X_1, \ldots, X_p) \mid_{t=0}\\ &= \frac{d}{dt}(\rho(e^{-tZ})\alpha(g)(\mathrm{Ad}_{e^{tZ}}X_1, \ldots, \mathrm{Ad}_{e^{tZ}}X_p)) \mid_{t=0}\\ &= -\dot{\rho}(Z)\alpha(X_1, \ldots, X_p) + \sum_{i=1}^{p}\alpha(X_1, \ldots, [Z, X_i], \ldots, X_p), \quad \forall Z \in \mathfrak{h}.\end{aligned}$$

The map $\bar{\partial}$ is the unique extension of $\bar{\partial} : \Gamma(G/H, V) \longrightarrow \Gamma(G/H, \mathcal{E}^{0,1} \otimes V)$, cf. Lemma 5.7. Write a local section α of $\mathcal{E}^{0,p} \otimes V$ as $\alpha = fs$, where f is a section of $\mathcal{E}^{0,p}$ and s a section of V, then one can use the known formulas for $\bar{\partial} f$ and $\bar{\partial} s$ to obtain the formula for $\bar{\partial}\alpha$:

$$\begin{aligned}&\bar{\partial}(fs)(X_0, \ldots, X_p)\\ &\quad = ((\bar{\partial} f)s)(X_0, \ldots, X_p) + ((-1)^p f \wedge \bar{\partial} s)(X_0, \ldots, X_p),\end{aligned}$$

where, by restricting the d of (5.33) to $\bar{\partial}$ and expressing it in term of Lie algebras, we have

$$\begin{aligned}&((\bar{\partial} f)s)(X_0, \ldots, X_p)\\ &\quad = \sum_{i=0}^{p} (-1)^i X_i f(X_0, \ldots, \hat{X}_i, \ldots, X_p)s\\ &\quad + \sum_{0 \leq i < j \leq p} (-1)^{i+j} f([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)s,\end{aligned}$$

and

$$\begin{aligned}&((-1)^p f \wedge \bar{\partial} s)(X_0, \ldots, X_p)\\ &\quad = (-1)^p \frac{1}{p!}\frac{1}{1!} \sum \operatorname{sgn}(\sigma) f \otimes \bar{\partial} s(X_{\sigma(0)}, \ldots, X_{\sigma(p-1)}, X_{\sigma(p)})\\ &\quad = (-1)^p \sum_{i=0}^{p} \frac{p!}{p!} (-1)^{p-i} f(X_0, \ldots \hat{X}_i, \ldots, X_p)(X_i + \dot{\rho}(X_i))s\\ &\quad = \sum_{i=0}^{p} (-1)^i f(X_0, \ldots \hat{X}_i, \ldots, X_p) X_i s\\ &\qquad + \sum_{i=0}^{p} (-1)^i \dot{\rho}(X_i)(f(X_0, \ldots \hat{X}_i, \ldots, X_p)s),\end{aligned}$$

where we have used

$$\begin{aligned}&\kappa \wedge \tau(X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q})\\ &\quad = \frac{1}{p!q!} \sum \operatorname{sgn}(\sigma) \kappa \otimes \tau(X_{\sigma(1)}, \ldots, X_{\sigma(p)}, X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}),\end{aligned} \tag{5.34}$$

for a p-form κ and a q-form τ, cf. e.g. [11].

We then obtain the result by noting $(X_i f)s + f X_i s = X_i(fs)$. Notice the result is independent of the decomposition $\alpha = fs$. $\square$

Proof of Proposition 5.42: Consider the following composition of maps on G/M,

$$\Gamma_{\mathfrak{r}}(G/M, \wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0) \xrightarrow{\widehat{\nabla}} \Gamma_{\mathfrak{r}}(G/M, \mathfrak{p}^* \otimes \wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0)$$

$$\xrightarrow{\pi} \Gamma_{\mathfrak{r}}(G/M, (\mathfrak{q}/\mathfrak{r})^* \otimes \wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0) \xrightarrow{\wedge} \Gamma_{\mathfrak{r}}(G/M, \wedge^{p+1}(\mathfrak{q}/\mathfrak{r})^* \otimes V_0),$$

where $\widehat{\nabla}$ is characterized by

$$\widehat{\nabla}\alpha(X; Y_1, \ldots, Y_p) = X\alpha(Y_1, \ldots, Y_p), \quad X \in \mathfrak{p} \text{ and all } Y_i \in \mathfrak{q},$$

and for a $(\mathfrak{r}, M)$-module E_0, we use $\Gamma_{\mathfrak{r}}(G/M, E_0)$ to denote $\text{Œ}(G/M, E)$. Note that π is induced by the well-defined map

$$(\mathfrak{q}/\mathfrak{r}) \longrightarrow \mathfrak{p}; \ X \longmapsto X^{\mathfrak{p}},$$

and $\wedge$ is induced by the wedge product of vector spaces $(\mathfrak{q}/\mathfrak{r})^*$ and $\wedge^p(\mathfrak{q}/\mathfrak{r})^*$.

The composition map $\wedge \circ \pi \circ \widehat{\nabla}$ is just D. Therefore $\tau_*^q D$ is the composition of three maps $\tau_*^q \widehat{\nabla}$, $\tau_*^q \pi$ and $\tau_*^q \wedge$.

As $\mathfrak{p}^*$ is associated to the bundle $\tau^* \mathcal{E}_X^1$, when we compute direct images, $\mathfrak{p}^*$ is just a passenger. Therefore

$$\tau_{*\mathfrak{r}}^q(\wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes \mathfrak{p}^* \otimes V_0) = \mathcal{E}_X^1 \otimes \tau_{*\mathfrak{r}}^q(\wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0).$$

To see that $\tau_*^q \widehat{\nabla}$ agrees with the first map ∇ of (5.31), one notes that $s \in H_{\mathfrak{r}}^q(G/M, \wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0)$ corresponds to an $\tilde{s} \in \Gamma(G/K, \tau_{*\mathfrak{r}}^q(\wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0))$. A section $\tilde{s}$ can be expressed as $\tilde{s} : \ G \longrightarrow W_0$, $W_0 = H_{\mathfrak{r}}^q(K/M, \wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0)$, satisfying $\tilde{s}(gk) = \tilde{\rho}(k)^{-1}\tilde{s}(g)$, for $k \in K$, where $\tilde{\rho}$ is the representation of K associated with the bundle $\tau_{*\mathfrak{r}}^q(\wedge^p(\mathfrak{q}/\mathfrak{r})^* \otimes V_0)$.

When $q = 0$, we can express elements of W as functions from K to $\wedge^p \mathfrak{q}^* \otimes V_0$ satisfying some conditions, cf. Section 5.3, then we have

$$\tilde{s}(gk)(k') = \tilde{\rho}(k)^{-1}\tilde{s}(g)(k') = \tilde{s}(g)(kk').$$

One can actually relate s and $\tilde{s}$ by $s(g) = \tilde{s}(g)(\text{id})$. The map $s \to \widehat{\nabla}s$ then just corresponds to $\tilde{s} \to \nabla\tilde{s}$ where $\nabla\tilde{s}(X) = X\tilde{s}$, $X \in \mathfrak{p}$, as $(Xs)(g) = \frac{d}{dt}s(ge^{tX}) \mid_{t=0}$ corresponds to $(X\tilde{s})(g) = \frac{d}{dt}\tilde{s}(ge^{tX}) \mid_{t=0}$. For general q the result is similar. Therefore the first map is simply the connection map ∇. □

Appendix 2: Realization of holomorphic Penrose transforms

Let X_R be a real analytic manifold which has a non-holomorphic twistor correspondence

$$Z_R \xleftarrow{\eta} F_R \xrightarrow{\tau} X_R,$$

with the four conditions of Section 5.3 satisfied. If X_R can be complexified to give a complex manifold X for which there exists a holomorphic twistor correspondence

$$Z \xleftarrow{\mu} F \xrightarrow{\nu} X,$$

with all four conditions of Section 1.1 satisfied, then we can consider the Penrose transform for this holomorphic correspondence. Since X_R is embedded in X as the fixed point set of the antiholomorphic involution associated with *the real structure* of X, (X has a real structure induced from X_R), we can restrict the solutions of field equation on X, which result from the holomorphic Penrose transform, to X_R and obtain isomorphisms between $H^s(Z, \mathcal{O}(V))$ and *real analytic* solutions of the field equation on X_R.

Remark: 1. If the field equation happens to admit only real analytic solutions, for example if the equation is elliptic (e.g. the Laplace equation on $\mathbb{R}^3$), then we can simply say that $H^p(Z, \mathcal{O}(V))$ is isomorphic to the solution space of that equation.

2. If Z_R is an open subset of Z, then all holomorphic vector bundles on Z can be restricted to give holomorphic vector bundles on Z_R, and the holomoprhic Penrose transform with a realization can be alternatively achieved by the non-holomorphic Penrose transform. However, as a holomorphic vector bundle on Z_R may not extend to a holomorphic vector bundle on Z, the non-holomorphic Penrose transform in general yields more results than the holomorphic one does.

Chapter 6

A Non-holomorphic Penrose Transform for Euclidean 3-Space

In this chapter we apply the non-holomorphic Penrose transform of Chapter 5 to our first example, where the target space X is Euclidean 3-space $\mathbb{R}^3$, cf. Whittaker [36] and Hitchin [15] for partial results. This is in fact a real form of the correspondence in Chapter 2, as is remarked in Appendix 2 of Chapter 5. Combining the results here with Chapter 2, we obtain immediately as a corollary that every smooth eigenfunction of the Laplacian on $\mathbb{R}^3$ is the restriction of a holomorphic eigenfunction of the holomorphic 'Laplace operator' on $\mathbb{C}^3$, and similarly for other equations.

6.1 Correspondence and set-up

The tangent bundle of a sphere can be written as

$$TS^2 = \{(\underline{n}, \underline{x}) \mid \underline{n}, \underline{x} \in \mathbb{R}^3, \underline{n} \cdot \underline{n} = 1, \underline{n} \cdot \underline{x} = 0\}.$$

One can identify TS^2 with the space of oriented lines in $\mathbb{R}^3$ by associating to $(\underline{n}, \underline{x}) \in TS^2$ the line through the point $\underline{x}$ with direction vector $\underline{n}$. Identifying TS^2 with the holomorphic tangent bundle $T\mathbb{CP}_1$, we shall again call it T and refer to it as the *minitwistor space for* $\mathbb{R}^3$, cf. [15].

Writing F for the space $F = \{(p, l) \mid p \in \mathbb{R}^3, l \text{ is an oriented line in } \mathbb{R}^3,\ p \in l\}$, one obtains a double fibration

$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ TS^2 & & \mathbb{R}^3, \end{tikzcd}$$

where η and τ are the obvious forgetful maps. The τ images of fibres of η are oriented lines in $\mathbb{R}^3$ and the fibre of τ at a point $x \in \mathbb{R}^3$ is biholomorphic to $\mathbb{CP}_1$, which corresponds to the sphere of directions at x.

Coordinates, groups and Lie algebras

Definition 6.1 *The group $ESU(2,\mathbb{C})$ is defined as follows*

$$ESU(2) = \{(A, B) \mid A \in SU(2), B \in \{2 \times 2 \textit{ Hermitian tracefree matrices}\}\},$$

where the group operation is: $(A, B) \circ (A', B') = (AA', AB'\bar{A}^t + B)$.

The coordinates on T, F and $\mathbb{R}^3$ shall be the same as that in Chapter 2, with X being Hermitian tracefree here. The group actions of $ESU(2)$ on these spaces are also the same as that of $ESL(2,\mathbb{C})$ in Chapter 2, with A^{-1} being replaced by $\bar{A}^t$.

The various Lie groups and Lie algebras we need are then listed below, where throughout this chapter, $s, t, u, v, w, z, \alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, $\theta, z_0 \in \mathbb{R}$:

$$G^{\mathbb{C}} = ESL(2,\mathbb{C}) \qquad \mathfrak{g} = \left\{\left(\begin{pmatrix} t & s \\ u & -t \end{pmatrix}, \begin{pmatrix} z & w \\ v & -z \end{pmatrix}\right)\right\}$$

$$L = \left\{\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} z_0 & 0 \\ 0 & -z_0 \end{pmatrix}\right)\right\} \qquad \mathfrak{l} = \left\{\left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}\right)\right\}$$

$$K = SU(2) \qquad \mathfrak{k} = \left\{\left(\begin{pmatrix} t & s \\ u & -t \end{pmatrix}, 0\right)\right\}$$

$$M = \left\{ \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, 0 \right) \right\} \qquad \mathfrak{m} = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, 0 \right) \right\}$$

$$Q = \left\{ \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix} \right) \right\} \quad \mathfrak{q} = \left\{ \left(\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}, \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix} \right) \right\}.$$

Also one has

$$\mathfrak{r} = \mathfrak{k} \cap \mathfrak{q} = \left\{ \left(\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}, 0 \right) \right\}.$$

Lemma 6.2 *The adjoint representations of $G^{\mathbb{C}} = ESL(2,\mathbb{C})$ and $\mathfrak{g} = \mathfrak{esl}(2,\mathbb{C})$ on $\mathfrak{esl}(2,\mathbb{C})$ are given by*

$$\begin{cases} Ad_{(A,B)}(X,Y) = (AXA^{-1}, [B, AXA^{-1}] + AYA^{-1}) \\ ad_{(X,Y)}(X',Y') = ([X,X'], [X,Y'] + [Y,X']), \end{cases}$$

where $(A,B) \in ESL(2,\mathbb{C})$, $(X,Y),(X',Y') \in \mathfrak{esl}(2,\mathbb{C})$.

Proof: Straightforward calculations. □

Therefore $\mathfrak{g}$ can be decomposed as $\mathfrak{k} \oplus \mathfrak{p}$, with the K–module $\mathfrak{p}$ being

$$\mathfrak{p} = \left\{ \left(0, \begin{pmatrix} z & w \\ v & -z \end{pmatrix} \right) \right\}.$$

One also has the desired property that $\mathfrak{q}/\mathfrak{r} = \mathfrak{p} \cap \mathfrak{q}$.

Homogeneous vector bundles

Lemma 6.3 *The line bundle $\mathcal{O}(n,\lambda)$ on T is associated to the $(\mathfrak{q}, L)$-module given by the following representations of L and $\mathfrak{q}$*

$$\begin{aligned} \rho\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} z_0 & 0 \\ 0 & -z_0 \end{pmatrix} \right) &= e^{-in\theta - \lambda z_0}, \\ \dot{\rho}\left(\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}, \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix} \right) &= -nt - \lambda z, \end{aligned}$$

where $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$.

Since $M = L \cap K = \left\{ \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, 0 \right) \right\}$, on pulling back the bundle $\mathcal{O}(n,\lambda)$ to F, the parameter λ is lost.

Definition 6.4 *Let* $\mathbb{E}(n)$ *denote the resulting pull-back bundle* $\eta^*\mathcal{O}(n,\lambda)$*, and* $\mathbb{C}(n)$ *its associated* $(\mathfrak{r}, M)$*-module.*

Remark: Although $\eta^*\mathcal{O}(n,\lambda) \cong \eta^*\mathcal{O}(n,\lambda')$, for $\lambda \neq \lambda'$, they come equipped with different $\bar{\partial}$ operators, making them different compatible vector bundles.

Definition 6.5 *Let* $\mathcal{E}^A$ *be the rank 2 bundle on* $\mathbb{R}^3$ *induced by the defining representation of* $K = SU(2)$*, and* $\nabla_A{}^B$ *be the spinor version of the Levi-Civita connection.*

Note $F = \mathbb{R}^3 \times \mathbb{P}_1$, and $\mathbb{E}(n)$ can be easily checked to be the sheaf of functions on F, of homogeneity n on $\mathbb{P}_1$. Then using Proposition 0.31, we obtain direct images of $\mathbb{E}(n)$ (use $\mathcal{E}^A \cong \mathcal{E}_A$ also).

6.2 Results

We will discuss the isomorphisms for $H^1(T, \mathcal{O}(n,\lambda))$ in three cases: $n \geq -1$, $n \leq -3$ and $n = -2$.

Proposition 6.6 *For* $n \geq -1$*, one has*

$$H^1(G/L, \mathcal{O}(n,\lambda)) \cong \frac{\left\{\nabla^{A(H}\psi_A{}^{BC\ldots E)} = \lambda\psi^{(BC\ldots H)}\right\}}{\left\{\nabla_A{}^{(B}\gamma^{C\ldots E)} + \lambda\epsilon_A{}^{(B}\gamma^{C\ldots E)}\right\}},$$

where $\gamma^{C\ldots E} \in \Gamma(\mathbb{R}^3, \odot^n\mathcal{E}^A)$*,* $\psi_A{}^{BC\ldots E} \in \Gamma(\mathbb{R}^3, \mathcal{E}_A \otimes \odot^{n+1}\mathcal{E}^A)$ *and* $\odot^{-1}\mathcal{E}^A$ *is taken to be vacuous.*

Proof: Consider the exact sequence of $(\mathfrak{r}, M)$-modules

$$\begin{array}{ccccccccc}
 & & w & \mapsto & \left(0, \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}\right) & & & & \\
 & & \Cap & & \Cap & & & & \\
0 & \to & \mathbb{C}(-2) & \to & \mathfrak{q}/\mathfrak{r} & \to & \mathbb{C} & \to & 0 \\
 & & & & \Cup & & \Cup & & \\
 & & & & \left(0, \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix}\right) & \mapsto & z. & &
\end{array}$$

This induces

$$0 \to \mathbb{E} \to \mathbb{E}(\mathcal{E}_\eta^{1,0}) \to \mathbb{E}(2) \to 0 \text{ on } F.$$

Tensoring it with $\mathbb{E}(n)$, we obtain the following short exact sequence

$$0 \to \mathbb{E}(n) \to \mathbb{E}(\mathcal{E}_\eta^{1,0}(n)) \to \mathbb{E}(n+2) \to 0,$$

which then, as $\mathbb{R}^3 = G/K$ and $K = SU(2)$ is semisimple, gives rise to

$$0 \to \odot^n \mathcal{E}^A \to \odot^{n+2}\mathcal{E}^A \oplus \odot^n \mathcal{E}^A \to \odot^{n+2}\mathcal{E}^A \to 0$$

on $\mathbb{R}^3$, where $\odot^n\mathcal{E}^A$ is taken to be vacuous if $n < 0$.

Since $\odot^{n+2}\mathcal{E}^A \oplus \odot^n\mathcal{E}^A \cong \mathcal{E}_A \otimes \odot^{n+1}\mathcal{E}^A$, we can write a typical element in $\Gamma(G/K, \tau_*\mathbb{E}(\mathcal{E}_\eta^{1,0}(n)))$ as $\psi_A{}^{\overbrace{B\ldots D}^{n+1}}$, where $\psi_A{}^{B\ldots D} = \psi_A{}^{(B\ldots D)}$.

Now as $\mathfrak{q}/\mathfrak{r} = \mathfrak{p} \cap \mathfrak{q}$, the map $\widetilde{\Phi} : \tau_*\mathbb{E}(n) \to \tau_*\mathbb{E}(\mathcal{E}_\eta^{1,0}(n))$ is induced by $\dot{\rho}\left(0, \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix}\right) = -\lambda z$ alone. We then have

$$\begin{array}{ccccc}
 & \phi^{C\ldots D} & \mapsto & -\nabla_A{}^{(B}\phi^{C\ldots D)} - \lambda\epsilon_A{}^{(B}\phi^{C\ldots D)} & \\
 & \Cap & & \Cap & \\
0 \to & \Gamma(\mathbb{R}^3, \odot^n\mathcal{E}^A) & \to & \Gamma(\mathbb{R}^3, \mathcal{E}_A \otimes \odot^{n+1}\mathcal{E}^A) & \to \Gamma(\mathbb{R}^3, \odot^{n+2}\mathcal{E}^A) \to 0 \\
 & & & \Cup & \Cup \\
 & & & \psi_A{}^{B\ldots D} & \mapsto \nabla^{A(F}\psi_A{}^{B\ldots D)} - \lambda\psi^{(F\ldots D)}.
\end{array}$$

The spectral sequence (5.27) then gives us the result. □

Proposition 6.7 *For $n \geq 1$, one has*

$$H^1(G/L, \mathcal{O}(-n-2, \lambda)) \cong \left\{\nabla_A{}^B\phi_{B\ldots E} = -\lambda\phi_{AC\ldots E}\right\},$$

where $\phi_{B\ldots E} \in \Gamma(\mathbb{R}^3, \odot^n\mathcal{E}_A)$.

Proof: The E_1 level of the spectral sequence (5.27) is

$$\begin{array}{|ccccc}
\Gamma(\mathbb{R}^3, \odot^n\mathcal{E}_A) & \xrightarrow{\widetilde{D}+\widetilde{\Phi}} & \Gamma(\mathbb{R}^3, \mathcal{E}_A \otimes \odot^{n-1}\mathcal{E}_A) & \to & \Gamma(\mathbb{R}^3, \odot^{n-2}\mathcal{E}_A) \\
0 & \longrightarrow & 0 & \to & 0, \\ \hline
\end{array}$$

where the map $\widetilde{D} + \widetilde{\Phi}$ is

$$\phi_{B\ldots D} \longrightarrow \nabla_{AB}\phi^B{}_{C\ldots D} - \lambda\phi_{AC\ldots D}. \qquad □$$

Proposition 6.8 *There is an isomorphism:*

$$H^1(G/L, \mathcal{O}(-2,\lambda)) \cong \{\Delta\phi = -2\lambda^2\phi\},$$

where $\Delta := \nabla^{AB}\nabla_{AB}$ *and* $\phi \in \Gamma(\mathbb{R}^3, \mathcal{E})$.

Proof: To say ω is a smooth section of $\mathcal{E}^{1,0}_\eta(n)$ is to say

$$\begin{aligned}
\left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \omega(gm) \right\rangle &= e^{in\theta}\left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \omega(g) \right\rangle; \\
\left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega(gm) \right\rangle &= e^{i(n+2)\theta}\left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega(g) \right\rangle,
\end{aligned}$$

where $m = \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, 0\right) \in M$ and $\langle\ ,\ \rangle$ denotes the usual pairing of a vector space and its dual. Note that we have used

$$\langle X, \omega(gm)\rangle = \rho^{-1}(m)\langle \mathrm{Ad}(m)X, \omega(g)\rangle, \quad X \in \mathfrak{q},$$

which is the integral form of (5.21) in the present context.

For ω to be compatible with the involutive structure, i.e. $\bar{\partial}_\tau\omega = 0$, we must in addition have, cf. (5.33),

$$\begin{aligned}
\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right)\left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \omega \right\rangle &= -2\left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega \right\rangle; \\
\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right)\left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega \right\rangle &= 0.
\end{aligned}$$

Now we can define an operator $\nabla' : Œ(\mathcal{E}^{1,0}_\eta(n)) \to Œ(\mathcal{E}^{1,0}_\eta(n))$ by

$$\begin{aligned}
\left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \nabla'\omega \right\rangle &= \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \omega \right\rangle \\
&\quad +4\left(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega \right\rangle; \\
\left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \nabla'\omega \right\rangle &= \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \omega \right\rangle \\
&\quad -\left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega \right\rangle.
\end{aligned}$$

Note that $\nabla'\omega$ is indeed an involutive section of $\mathcal{E}^{1,0}_\eta(n)$.

Now consider the following diagram

$$\begin{array}{ccccccc}
0 \to & Œ(n) & \xrightarrow{\mu} & Œ(\mathcal{E}^{1,0}_\eta(n)) & \xrightarrow{\nu} & Œ(n+2) & \to 0 \\
& \| & & \downarrow \nabla' - \lambda & & \downarrow \Box' & \\
\to & Œ(n) & \xrightarrow{\partial} & Œ(\mathcal{E}^{1,0}_\eta(n)) & \xrightarrow{\partial} & Œ(n+2) & \to 0.
\end{array}$$

The map μ is characterized by

$$\left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \mu s \right\rangle = s, \quad \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \mu s \right\rangle = 0,$$

where s is a section of $\mathbb{C}\!\mathrm{E}(n)$.

The first ∂ is given, cf. (5.24), by

$$\begin{aligned} \left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \partial s \right\rangle &= \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) s - \lambda s; \\ \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \partial s \right\rangle &= \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) s. \end{aligned}$$

The map ν is

$$\nu(\omega) = \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega \right\rangle.$$

The second ∂ sends Ψ, sections of $\mathbb{C}\!\mathrm{E}(\mathcal{E}^{1,0}_\eta(n))$, to

$$\begin{aligned} &\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \Psi \right\rangle - \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \Psi \right\rangle \\ &\qquad + \lambda \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \Psi \right\rangle. \end{aligned}$$

One can see that the first square in the last diagram commutes. To make the second square commute, $\Box' \nu(\omega)$ must equal $\partial(\nabla' - \lambda)\omega$. Here we have that $\partial(\nabla' - \lambda)$ sends ω to

$$\begin{aligned} &\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \nabla'\omega \right\rangle - \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \nabla'\omega \right\rangle \\ &+\lambda \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \nabla'\omega \right\rangle - \lambda \left[\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \left\langle \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right), \omega \right\rangle \right. \\ &\left. -\left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega \right\rangle + \lambda \left\langle \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \omega \right\rangle \right] \\ = \;& 4\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \left(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \nu(\omega) + \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \nu(\omega) \\ &-\lambda^2 \nu(\omega). \end{aligned}$$

When $n = -2$, the zeroth direct image of $\Box'$ can be identified with

$$f \longmapsto (-\frac{1}{2}\Delta - \lambda^2) f, \quad f \in \Gamma(\mathbb{R}^3, \mathcal{E}_{\mathbb{R}^3}),$$

where $\Delta := \nabla^{AB}\nabla_{AB} = -2\nabla^a\nabla_a$. See Lemma 6.9 for details. Then by arguments similar to that in the proof of Proposition 2.12 we obtain the isomorphism. $\Box$

Lemma 6.9 *One can identify the zeroth direct image of* $\Box' : \mathbb{E} \longrightarrow \mathbb{E}$ *with*

$$-\frac{1}{2}\Delta - \lambda^2 : \mathcal{E}_{\mathbb{R}^3} \longrightarrow \mathcal{E}_{\mathbb{R}^3}.$$

Proof: Let X, Y, Z be the vector fields generated by $\left(0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$, $\left(0, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right)$ and $\left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ respectively. Then we have

$$\begin{aligned}
&\left(4\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\left(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) + \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right)\nu(\omega) - \lambda^2\nu(\omega) \\
= \;& 4(\tfrac{1}{2}X - \tfrac{i}{2}Y)(\tfrac{1}{2}X + \tfrac{i}{2}Y)\nu(\omega) + Z^2\nu(\omega) - \lambda^2\nu(\omega) \\
= \;& (X^2 + Y^2 + Z^2)\nu(\omega) - \lambda^2\nu(\omega).
\end{aligned}$$

Note here X, Y, Z are vector fields on G, and one can compute the corresponding vectors on $\mathbb{R}^3$ at $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^3$ to be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively. Therefore the corresponding map between direct images is

$$f \longmapsto (\partial_x^2 + \partial_y^2 + \partial_z^2)f - \lambda^2 f = -\frac{1}{2}\nabla^{AB}\nabla_{AB}f - \lambda^2 f. \qquad \Box$$

By virtue of comparing the results of this section to that of Chapter 2, one has the following corollary.

Corollary 6.10 *Every smooth eigenfunction of the Laplacian on* $\mathbb{R}^3$ *extends to a holomorphic eigenfunction of the holomorphic Laplacian on* $\mathbb{C}^3$. *The same applies to eigenfunctions of the Dirac operator, and all other systems of equations applying as image of a Penrose transform in Propositions 6.6 and 6.7.*

Proof: For the Laplace case, the holomorphic Penrose transform of Chapter 2 provides a map $\mathcal{P}$ from $H^1(T, \mathcal{O}(-2, \lambda))$ to holomorphic solutions of $\Delta\phi = -2\lambda^2\phi$ on $\mathbb{C}^3$. On restricting to $\mathbb{R}^3 \hookrightarrow \mathbb{C}^3$, this agrees with the non-holomorphic Penrose transform of this section. The other cases are analogous. $\Box$

Remark: One knows that solutions are analytic by ellipticity, but the result here is stronger.

Chapter 7

A Non-holomorphic Penrose Transform for Hyperbolic 3-Space

A non-holomorphic Penrose transform for hyperbolic 3-space H, considered as $SO_\circ(3,1)/SO(3)$, was first considered in [3]. In this chapter, by considering a different symmetry group $SL(2,\mathbb{C})$, we are able to obtain additional results over [3] because fields on H taking values in spin representations are included. This Penrose transform is a real form of the case considered in Chapter 3. However, we obtain more results here as the twistor space of H considered is only an open orbit of the twistor space of Chapter 3. In particular, in classifying homogeneous line bundles over the twistor space, one has one discrete parameter n and one continuous parameter λ here, in contrast to two discrete parameters n, λ in Chapter 3. Combining the results here with Chapter 3, we obtain as a corollary that every smooth eigenfunction of the Laplacian on H extends to a holomorphic eigenfunction of the holomorphic Laplacian on an open subset $\widehat{\mathbb{H}}$ (to be defined) of $\mathbb{H}$, and similarly for other equations.

7.1 Correspondence and set-up

One model for hyperbolic 3-space is the hypersurface

$$H = \{(t,x,y,z) \in \mathbb{C} \mid t^2 - x^2 - y^2 - z^2 = 1,\ t > 0\}$$

in $\mathbb{R}^4$ with the metric obtained by restricting the metric $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$

on $\mathbb{R}^4$.

A geodesic in H is the intersection of H with a 2-plane in $\mathbb{R}^4$ (through the origin). Each of such planes intersects N, the null cone of the origin, at two distinct generators, which can be used to characterize the corresponding geodesic, cf. [3]. An oriented geodesic is then an ordered pair of distinct generators of N. As the space of generators of N is isomorphic to $S^2 \cong \mathbb{P}_1$, one concludes that the space of oriented geodesics in H is isomorphic to $(\mathbb{P}_1 \times \mathbb{P}_1)\backslash\Delta$, which shall be referred to as the *minitwistor space for H*, where Δ is the anti-diagonal in $\mathbb{P}_1 \times \mathbb{P}_1$.

We can introduce the correspondence space F to be

$$F = \{(x, l) \mid x \in H,\ l \text{ is an oriented geodesic in } H, \text{ and } x \in l\},$$

which is just $H \times \mathbb{P}_1$. Then one can set up a correspondence

$$\begin{tikzcd} & H \times \mathbb{P}_1 \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ (\mathbb{P}_1 \times \mathbb{P}_1)\backslash\Delta & & H, \end{tikzcd}$$

where η and τ are the obvious forgetful maps. The τ images of fibres of η are oriented geodesics in H and the fibre of τ at point $x \in H$ is biholomorphic to $\mathbb{CP}_1$, corresponding to the sphere of directions at x.

Coordinates, groups and Lie algebras

Think of elements of $\mathbb{R}^4$ as 2×2 Hermitian matrices

$$X = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}, \qquad x, y, z, t \in \mathbb{R}.$$

Then H is identified with the determinant 1 hypersurface of $\mathbb{R}^4$. We introduce coordinates on $\mathbb{P}_1 \times \mathbb{P}_1\backslash\Delta$, F and H as that in Chapter 3, with $X^{AA'}$ being Hermitian matrices. Note the anti-diagonal Δ is the image of $\mathbb{P}_1 \hookrightarrow \mathbb{P}_1 \times \mathbb{P}_1$ given by

$$[\pi_A] \longrightarrow ([\pi_A], [\bar{\pi}^{A'}]).$$

The $g \in SL(2,\mathbb{C})$ acts on $\mathbb{P}_1 \times \mathbb{P}_1 \backslash \Delta$, F and H respectively by:

$$\begin{aligned} (g,\bar{g})([\pi],[\xi]) &= (g^{t^{-1}}[\pi], \bar{g}[\xi]) && \text{on } Q, \\ (g,\bar{g})([X],[\pi]) &= (g[X]\bar{g}^t, g^{t^{-1}}[\pi]) && \text{on } F, \\ (g,\bar{g})[X] &= g[X]\bar{g}^t && \text{on } H. \end{aligned}$$

We can list the various Lie groups and Lie algebras: ($\theta \in \mathbb{R}$, $0 \neq \alpha, \alpha' \in \mathbb{C}$ and $s,t,u,v,w,z,\beta,\beta' \in \mathbb{C}$)

$$G^{\mathbb{C}} = SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \qquad \mathfrak{g} = \left\{ \left(\begin{pmatrix} t & s \\ u & -t \end{pmatrix}, \begin{pmatrix} z & w \\ v & -z \end{pmatrix} \right) \right\}$$

$$L = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\} \qquad \mathfrak{l} = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \right) \right\}$$

$$K = SU(2) \qquad \mathfrak{k} = \left\{ \left(\begin{pmatrix} t & s \\ u & -t \end{pmatrix}, \begin{pmatrix} -t & -u \\ -s & t \end{pmatrix} \right) \right\}$$

$$M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\} \qquad \mathfrak{m} = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \begin{pmatrix} -t & 0 \\ 0 & t \end{pmatrix} \right) \right\}$$

$$Q = \left\{ \left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \alpha' & 0 \\ \beta' & \alpha'^{-1} \end{pmatrix} \right) \right\} \qquad \mathfrak{q} = \left\{ \left(\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}, \begin{pmatrix} z & 0 \\ v & -z \end{pmatrix} \right) \right\}.$$

One also has

$$\mathfrak{r} = \mathfrak{k} \cap \mathfrak{q} = \left\{ \left(\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}, \begin{pmatrix} -t & 0 \\ -s & t \end{pmatrix} \right) \right\}.$$

Lemma 7.1 *The adjoint representations of $G^{\mathbb{C}} = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ on $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ are given by*

$$\begin{cases} Ad_{(A,B)}(X,Y) = (AXA^{-1}, BYB^{-1}) \\ ad_{(X,Y)}(X',Y') = ([X,X'],[Y,Y']), \end{cases}$$

where $(A,B) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, $(X,Y),(X',Y') \in \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

Proof: Standard calculation. □

Therefore $\mathfrak{g}$ can be decomposed as $\mathfrak{k} \oplus \mathfrak{p}$, where the K-module $\mathfrak{p}$ is given by

$$\mathfrak{p} = \left\{ \left(\begin{pmatrix} z & v \\ w & -z \end{pmatrix}, \begin{pmatrix} z & w \\ v & -z \end{pmatrix} \right) \right\}.$$

One also has the desired property $\mathfrak{q}/\mathfrak{r} = \mathfrak{p} \cap \mathfrak{q}$.

Homogeneous vector bundles

Definition 7.2 *The holomorphic line bundle $\mathcal{O}(n, \lambda)$, where $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, on $\mathbb{P}_1 \times \mathbb{P}_1 \backslash \Delta$ is associated to the $(\mathfrak{q}, L)$-module given by:*

$$\begin{aligned} \rho\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) &= \left(\tfrac{\alpha}{\bar{\alpha}}\right)^{-\frac{n}{2}} (\alpha\bar{\alpha})^{-\lambda}; \\ \dot{\rho}\left(\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}, \begin{pmatrix} z & 0 \\ v & -z \end{pmatrix}\right) &= -(\tfrac{n}{2} + \lambda)t + (\tfrac{n}{2} - \lambda)z. \end{aligned}$$

Lemma 7.3 *The bundle $\mathcal{O}(n, \lambda)$ on $\mathbb{P}_1 \times \mathbb{P}_1 \backslash \Delta$ can be extended to a holomorphic line bundle on $\mathbb{P}_1 \times \mathbb{P}_1$ if and only if $\frac{n}{2} + \lambda$ and $\frac{n}{2} - \lambda$ are both integers.*

Therefore restricting the holomorphic Penrose transform for $\mathbb{H}$ considered in Chapter 3 to H does not give all the results that we can get from the non-holomorphic Penrose transform of this chapter.

Since $M = L \cap K = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \ \theta \in \mathbb{R} \right\}$, on pulling back the bundle $\mathcal{O}(n, \lambda)$ to F, the parameter λ is lost.

Definition 7.4 *Let $\mathbb{E}(n)$ denote the resulting pull-back bundle $\eta^* \mathcal{O}(n, \lambda)$, and $\mathbb{C}(n)$ be the corresponding $(\mathfrak{r}, M)$-module.*

Definition 7.5 *Let $\mathcal{E}^A$ be the rank 2 bundle on H induced by the defining representation of $K = SU(2)$, and ∇_{AB} be the spinor version of the Levi-Civita connection.*

As in the $\mathbb{R}^3$ case, using Proposition 0.31, we obtain direct images of $\mathbb{E}(n)$.

7.2 Results

We shall then discuss $H^1(G/L, \mathcal{O}(n,\lambda))$ in three cases: $n \geq -1$, $n \leq -3$ and $n = -2$.

Proposition 7.6 *For $n \geq -1$, one has*

$$H^1(G/L, \mathcal{O}(n,\lambda)) \cong \frac{\left\{\nabla^{A(H}\psi_A{}^{BC\ldots E)} = \lambda\psi^{(BC\ldots H)}\right\}}{\left\{\nabla_A{}^{(B}\gamma^{C\ldots E)} + \lambda\epsilon_A{}^{(B}\gamma^{C\ldots E)}\right\}},$$

where $\gamma^{C\ldots E} \in \Gamma(H, \odot^n\mathcal{E}^A)$, $\psi_A{}^{BC\ldots E} \in \Gamma(H, \mathcal{E}_A \otimes \odot^{n+1}\mathcal{E}^A)$ and $\odot^{-1}\mathcal{E}^A$ is taken to be vacuous.

Proof: Consider the exact sequence

$$\begin{array}{ccccccccc}
 & & v & \mapsto & \left(\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}\right) & & & & \\
 & & \Cap & & \Cap & & & & \\
0 & \to & \mathbb{C}(-2) & \to & \mathfrak{q}/\mathfrak{r} & \to & \mathbb{C} & \to & 0 \\
 & & & & \Cup & & \Cup & & \\
 & & & & \left(\begin{pmatrix} z & v \\ 0 & -z \end{pmatrix}, \begin{pmatrix} z & 0 \\ v & -z \end{pmatrix}\right) & \mapsto & z. & &
\end{array}$$

It induces

$$0 \to \text{Œ} \to \text{Œ}(\mathcal{E}_\eta^{1,0}) \to \text{Œ}(2) \to 0.$$

Tensoring it with $\text{Œ}(n)$, we obtain the short exact sequence

$$0 \to \text{Œ}(n) \to \text{Œ}(\mathcal{E}_\eta^{1,0}(n)) \to \text{Œ}(n+2) \to 0$$

which gives rise to

$$0 \to \odot^n\mathcal{E}^A \to \odot^{n+2}\mathcal{E}^A \oplus \odot^n\mathcal{E}^A \to \odot^{n+2}\mathcal{E}^A \to 0 \text{ on } H.$$

Since $\odot^{n+2}\mathcal{E}^A \oplus \odot^n\mathcal{E}^A \cong \mathcal{E}_A \otimes \odot^{n+1}\mathcal{E}^A$, a typical element in $\Gamma(H, \tau_*\text{Œ}(\mathcal{E}_\eta^{1,0}(n)))$ is $\psi_A{}^{\overbrace{B\ldots D}^{n+1}}$, where $\psi_A{}^{B\ldots D} = \psi_A{}^{(B\ldots D)}$. Now, as $\mathfrak{q}/\mathfrak{r} = \mathfrak{p} \cap \mathfrak{q}$, $\widetilde{\Phi} : \tau_*\text{Œ}(n) \to$

$\tau_*Œ(\mathcal{E}_\eta^{1,0}(n))$ is induced by

$$\dot\rho\left(\begin{pmatrix} z & v \\ 0 & -z \end{pmatrix}, \begin{pmatrix} z & 0 \\ v & -z \end{pmatrix}\right) = -(\frac{n}{2}+\lambda)z + (\frac{n}{2}-\lambda)z = -2\lambda z$$

alone. We then have

$$\begin{array}{ccccc} \phi^{C\ldots D} \mapsto -2\nabla_A{}^{(B}\phi^{C\ldots D)} - 2\lambda\epsilon_A{}^{(B}\phi^{C\ldots D)} & & & & \\ \Cap & & \Cap & & \\ 0 \to \Gamma(H, \odot^n\mathcal{E}^A) \to & & \Gamma(H, \mathcal{E}_A \otimes \odot^{n+1}\mathcal{E}^A) & & \to \Gamma(H, \odot^{n+2}\mathcal{E}^A) \to 0. \\ & & \Cup & & \Cup \\ & & \psi_A{}^{B\ldots D} & & \mapsto 2\nabla^{A(F}\psi_A{}^{B\ldots D)} - 2\lambda\psi^{(F\ldots D)}. \end{array}$$

Notice the appearance of the 2's in front of ∇_{AB} 's. They come from relating ∇ of (5.30) with ∇_{AB} by explicitly computing Xs, cf. also the proof of Lemma 7.9. □

Proposition 7.7 *For $n \geq 1$, one has*

$$H^1(G/L, \mathcal{O}(-n-2,\lambda)) \cong \left\{\nabla_A{}^B\phi_{B\ldots E} = -\lambda\phi_{AC\ldots E}\right\},$$

where $\phi_{B\ldots E} \in \Gamma(H, \odot^n\mathcal{E}_A)$.

Proof: The E_1 level of the spectral sequence (5.27) is

$$\begin{array}{|ccccc} \Gamma(H,\odot^n\mathcal{E}_A) & \xrightarrow{\widetilde{D}+\widetilde{\Phi}} & \Gamma(H,\odot^{n-1}\mathcal{E}_A\otimes\mathcal{E}_A) & \to & \Gamma(H,\odot^{n-2}\mathcal{E}_A) \\ 0 & \longrightarrow & 0 & \to & 0, \\ \hline \end{array}$$

where the map $\widetilde{D}+\widetilde{\Phi}$ is

$$\phi_{B\ldots D} \longrightarrow 2\nabla_{AB}\phi^B{}_{C\ldots D} - 2\lambda\phi_{AC\ldots D}.$$

Therefore we have the result. □

Proposition 7.8 *There is an isomorphism:*

$$H^1(G/L, \mathcal{O}(-2,\lambda)) \cong \{\Delta\phi = -2(\lambda^2-1)\phi\},$$

where $\Delta := \nabla^{AB}\nabla_{AB}$ and $\phi \in \Gamma(H, \mathcal{E})$.

Proof: Recall that to say ω is a smooth section of $\mathcal{E}^{1,0}_{\eta}(n)$ is to say

$$\begin{aligned}\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\omega(gm)\right\rangle &= e^{in\theta}\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\omega(g)\right\rangle;\\ \left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\omega(gm)\right\rangle &= e^{i(n+2)\theta}\left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\omega(g)\right\rangle.\end{aligned}$$

where $m = \begin{pmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{pmatrix} \in M$.

For ω to be compatible with the involutive structure, we need

$$\begin{aligned}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\-1&0\end{pmatrix}\right)\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\omega\right\rangle &= -2\left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\omega\right\rangle;\\ \left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\-1&0\end{pmatrix}\right)\left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\omega\right\rangle &= 0.\end{aligned}$$

Now define an operator $\nabla' : \mathbb{C}\!\mathrm{E}(\mathcal{E}^{1,0}_{\eta}(n)) \to \mathbb{C}\!\mathrm{E}(\mathcal{E}^{1,0}_{\eta}(n))$ by

$$\begin{aligned}\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\nabla'\omega\right\rangle &= \left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\omega\right\rangle\\ &\quad +4\left(\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\omega\right\rangle;\\ \left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\nabla'\omega\right\rangle &= \left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\omega\right\rangle\\ &\quad -\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\omega\right\rangle.\end{aligned}$$

Consider the following diagram

$$\begin{array}{ccccccc} 0\to & \mathbb{C}\!\mathrm{E}(n) & \xrightarrow{\mu} & \mathbb{C}\!\mathrm{E}(\mathcal{E}^{1,0}_{\eta}(n)) & \xrightarrow{\nu} & \mathbb{C}\!\mathrm{E}(n+2) & \to 0\\ & \| & & \downarrow \nabla'-2\lambda & & \downarrow \Box' & \\ \to & \mathbb{C}\!\mathrm{E}(n) & \xrightarrow{\partial} & \mathbb{C}\!\mathrm{E}(\mathcal{E}^{1,0}_{\eta}(n)) & \xrightarrow{\partial} & \mathbb{C}\!\mathrm{E}(n+2) & \to 0.\end{array}$$

The map μ is characterized by

$$\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\mu s\right\rangle = s,\quad \left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\mu s\right\rangle = 0,$$

where s is a section of $\mathbb{C}\!\mathrm{E}(n)$.

Similarly the first ∂ is given by

$$\begin{aligned}\left\langle\left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right),\partial s\right\rangle &= \left(\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)s - 2\lambda s;\\ \left\langle\left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),\partial s\right\rangle &= \left(\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right)s.\end{aligned}$$

The map ν is

$$\nu(\omega) = \left\langle \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \omega \right\rangle .$$

The second ∂ sends Ψ, sections of $\mathbb{E}(\mathcal{E}_\eta^{1,0}(n))$ to

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \left\langle \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \Psi \right\rangle$$

$$-\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \left\langle \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \Psi \right\rangle + 2\lambda \left\langle \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \Psi \right\rangle .$$

One can see that the first square in the last diagram commutes. To make the second square commute, $\Box'\nu(\omega)$ must equal $\partial(\nabla' - 2\lambda)\omega$. Now $\partial(\nabla' - 2\lambda)$ sends ω to

$$\begin{aligned}
& \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \left\langle \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \nabla'\omega \right\rangle \\
& + \left(-\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) + 2\lambda \right) \left\langle \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \nabla'\omega \right\rangle \\
& -2\lambda \left[\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \left\langle \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \omega \right\rangle \right. \\
& \left. -\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \left\langle \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \omega \right\rangle + 2\lambda \left\langle \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \omega \right\rangle \right] \\
= \; & \left[\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right] \left\langle \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \omega \right\rangle \\
& +4 \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \nu(\omega) \\
& + \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \nu(\omega) - 4\lambda^2 \nu(\omega) \\
= \; & 4\nu(\omega) - 4\lambda^2\nu(\omega) + 4 \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \nu(\omega) \\
& + \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \nu(\omega).
\end{aligned}$$

When $n = -2$, the zeroth direct image of $\Box'$ can then be identified with

$$4 - 4\lambda^2 - 2\triangle : \mathcal{E}_H \longrightarrow \mathcal{E}_H,$$

see Lemma 7.9 to come. Then by arguments similar to that in the proof of Proposition 2.12 we obtain the isomorphism. ∎

Lemma 7.9 *The zeroth direct image of* $\square' : \mathcal{Œ} \longrightarrow \mathcal{Œ}$ *can be identified with*

$$-2\Delta + 4 - 4\lambda^2 : \mathcal{E}_H \longrightarrow \mathcal{E}_H.$$

Proof: Let X,Y and Z be the vector fields generated by $\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$, $\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$ and $\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ respectively. Then we have

$$
\begin{aligned}
& 4\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\nu(\omega) \\
& \quad + \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\nu(\omega) + (4 - 4\lambda^2)\nu(\omega) \\
= \; & 4(\tfrac{1}{2}X - \tfrac{i}{2}Y)(\tfrac{1}{2}X + \tfrac{i}{2}Y)\nu(\omega) + Z^2\nu(\omega) + (4 - 4\lambda^2)\nu(\omega) \\
= \; & X^2\nu(\omega) + 2\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)\nu(\omega) + Y^2\nu(\omega) + Z^2\nu(\omega) + (4 - 4\lambda^2)\nu(\omega) \\
= \; & (X^2 + Y^2 + Z^2)\nu(\omega) + 2\left\langle\left[\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\right], \omega\right\rangle \\
& \quad -2[-(\tfrac{-2}{2} + \lambda) - (\lambda - \tfrac{-2}{2})(-1)]\nu(\omega) + (4 - 4\lambda^2)\nu(\omega) \\
= \; & (X^2 + Y^2 + Z^2)\nu(\omega) + 4\nu(\omega) - 4\nu(\omega) + (4 - 4\lambda^2)\nu(\omega) \\
= \; & (X^2 + Y^2 + Z^2)\nu(\omega) + (4 - 4\lambda^2)\nu(\omega).
\end{aligned}
$$

Note here X, Y, Z are vector fields on G. One can compute the corresponding vectors on H at $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H$. They are $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ respectively. Therefore we obtain that the corresponding direct image map is

$$f \longmapsto 4(\partial_x^2 + \partial_y^2 + \partial_z^2)f + (4 - 4\lambda^2)f = -2\nabla^{AB}\nabla_{AB}f + (4 - 4\lambda^2)f,$$

where $f \in \Gamma(H, \mathcal{E}_H)$. $\square$

Definition 7.10 *Let* $\widehat{\mathbb{H}}$ *denote the open subset of* $\mathbb{H}$ *defined by*

$$\widehat{\mathbb{H}} := \{X^{AA'} \in \mathbb{H} \mid X^{AA'}\pi_A\bar{\pi}_{A'} \neq 0, \forall \pi_A \neq 0\}.$$

Note that $\widehat{\mathbb{H}}$ consists of those points in $\mathbb{H}$ such that the corresponding conic in the sense of Chapter 3 does not intersect the anti-diagonal Δ.

Note also that all the differential operators on H involved in describing the right hand side of Penrose transform extend to holomorphic operators on $\mathbb{H}$, and hence, in particular, to $\widehat{\mathbb{H}}$.

Corollary 7.11 *Every smooth eigenfunction of the Laplacian on H extends to a holomorphic eigenfunction of the holomorphic Laplacian on $\widehat{\mathbb{H}}$. The same applies to eigenfunctions of the Dirac operator, and all other systems of equations applying as image of a Penrose transform in Propositions 7.6 and 7.7.*

Proof: For the Laplace case, the holomorphic Penrose transform essentially as in Chapter 3 provides a map $\mathcal{P}$ from $H^1((\mathbb{P}_1 \times \mathbb{P}_1)\backslash\Delta, \mathcal{O}(-2, \lambda))$ to holomorphic solutions of $\Delta\phi = -2(\lambda^2 - 1)\phi$ on $\widehat{\mathbb{H}}$, where λ is a complex value parameter. It is easy to see that on restriction to $H \hookrightarrow \widehat{\mathbb{H}}$, this agrees with the non-holomorphic Penrose transform of this section. The other cases are analogous. $\square$

Remark: One knows that solutions are analytic by ellipticity, but we are obtaining a stronger result.

Chapter 8

A Penrose Transform for $\mathbb{R}^5$

In this chapter we apply the non-holomorphic Penrose transform to a case which has not been considered in the literature before, yet deserves some attention: The Euclidean 5-space $\mathbb{R}^5$, considered as the space of tracefree symmetric 3×3 matrices, has a natural twistor corresponaence, see Section 8.2, and one can consider its associated Penrose transform, see Section 8.3. A corresponding holomorphic twistor correspondence for the complexefied space $\mathbb{C}^5$ is in fact a 'flat' example of a more general twistor correspondence to be considered in Chapter 10, and is introduced in Section 8.1. In an Appendix, an alternative approach where $\mathbb{C}^5$ is given the structure of $\odot^4\mathbb{C}^2$ is considered.

8.1 Holomorphic twistor correspondence

The correspondence

By considering the reduction of $SO(5,\mathbb{C})$ on $\mathbb{C}^5$ to a subgroup $SO(3,\mathbb{C}) \hookrightarrow SO(5,\mathbb{C})$, we can give $\mathbb{C}^5$ the structure of $\odot^2_\circ\mathbb{C}^3$, the space of tracefree symmetric 3×3 complex matrices with $\mathbb{C}^3$ being equipped with its standard symmetric bilinear form. In terms of matrices, the inner product of $X, X' \in \mathbb{C}^5$ is then given by

$$\langle X, X' \rangle = \frac{1}{2}\mathrm{trace}(XX').$$

Definition 8.1 *Let $\mathbb{C}^3_\circ$ be the space of null vectors in $\mathbb{C}^3$. A vector in $\mathbb{C}^5$ is quadruple null if it is of the form ZZ^t, where $Z \in \mathbb{C}^3_\circ$. A quadruple null hyperplane is a hyperplane whose normal is quadruple null everywhere.*

See Appendix for an interpretation of quadruple nullness in terms of 2-spinors, where the usage of the term is explained.

The space of quadruple-null hyperplanes in $\mathbb{C}^5$ can be identified with

$$T := \{(Z, \xi) \mid Z \in \mathbb{C}^3_\circ, \xi \in \mathbb{C}\} \,/\, \sim,$$

where $(Z, \xi) \sim (\lambda Z, \lambda^2 \xi)$ for $\lambda \in \mathbb{C}^*$, and the 4-null hyperplane associated with (Z, ξ) is given by $\{X \mid Z^t X Z = \xi\}$. The space T can also be identified with the total space of $\mathcal{O}(4) \longrightarrow \mathbb{CP}_1$.

Let $\mathbb{F}$ be the space

$$\mathbb{F} = \{(\Sigma, x) \mid x \in \mathbb{C}^5, \Sigma \text{ is a quadruple-null hyperplane in } \mathbb{C}^5, x \in \Sigma\}.$$

As $\mathbb{F}$ is isomorphic to $\mathbb{C}^5 \times \mathbb{P}(\mathbb{C}^3_\circ)$, we have the following correspondence

$$\begin{tikzcd}
& \mathbb{C}^5 \times \mathbb{P}_1(\mathbb{C}^3_\circ) \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\
T & & \mathbb{C}^5,
\end{tikzcd}$$

where μ, ν are the obvious forgetful maps. For $p \in T$, $\nu \circ \mu^{-1}(p)$ is a 4-null hyperplane in $\mathbb{C}^5$, while for $X \in \mathbb{C}^5$, $\mu \circ \nu^{-1}(X)$ is a section of $\mathcal{O}(4)$.

Coordinates and groups

An element $X \in \mathbb{C}^5 = \odot^2_\circ \mathbb{C}^3$ will in general be written as

$$X = \begin{pmatrix} \frac{1}{\sqrt{3}}x + z & y & u \\ y & \frac{1}{\sqrt{3}}x - z & v \\ u & v & \frac{-2}{\sqrt{3}}x \end{pmatrix}, \tag{8.1}$$

where $x, y, z, u, v \in \mathbb{C}$. The standard metric on $\mathbb{C}^5$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2 + du^2 + dv^2.$$

Nonetheless, we shall also introduce an alternative coordinate:

$$X = \begin{pmatrix} x_1 - ix_2 + ix_5 & x_2 + x_5 & x_3 + x_4 \\ x_2 + x_5 & x_1 + ix_2 - ix_5 & ix_3 - ix_4 \\ x_3 + x_4 & ix_3 - ix_4 & -2x_1 \end{pmatrix}, \tag{8.2}$$

where $x_1, x_2, x_3, x_4, x_5 \in \mathbb{C}$. In terms of this coordinate the standard metric is

$$ds^2 = 3dx_1^2 + 4dx_2dx_5 + 4dx_3dx_4.$$

Definition 8.2 *The group $ESO(3, \mathbb{C})$ is defined by*

$$ESO(3, \mathbb{C}) = \{(A, B) \mid A \in SO(3, \mathbb{C}), B \in \mathbb{C}^5\},$$

with group composition rule

$$(A_2, B_2) \circ (A_1, B_1) = (A_2A_1, A_2B_1A_2^t + B_2). \tag{8.3}$$

The group $ESO(3, \mathbb{C})$ acts transitively on $\mathbb{C}^5$, $\mathbb{F}$ and T by

$$\begin{aligned} (A, B)X &= AXA^t + B && \text{on } \mathbb{C}^5, \\ (A, B)(X, Z) &= (AXA^t + B, AZ) && \text{on } \mathbb{F}, \\ (A, B)(Z, \xi) &= (AZ, \xi + Z^tA^tBAZ) && \text{on } T. \end{aligned}$$

One then obtains

$$T = ESO(3, \mathbb{C})/Q, \quad \mathbb{F} = ESO(3, \mathbb{C})/R, \quad \mathbb{C}^5 = ESO(3, \mathbb{C})/SO(3, \mathbb{C}),$$

where

$$Q = \left\{ \left(\begin{pmatrix} r & p & e \\ q & r + (p+q)i & f \\ g & gi & k \end{pmatrix}, \begin{pmatrix} a - ib & b & c + d \\ b & a + ib & ic - id \\ c + d & ic - id & -2a \end{pmatrix} \right) \in ESO(3, \mathbb{C}) \right\}, \tag{8.4}$$

where $a, b, c, d, e, f, g, k, p, q, r \in \mathbb{C}$, and $R = Q \cap SO(3, \mathbb{C})$, where $SO(3, \mathbb{C})$ is a natural subgroup of $ESO(3, \mathbb{C})$.

8.2 Non-holomorphic twistor correspondence

By reducing $SO(5,\mathbb{R})$ on $\mathbb{R}^5$ to $SO(3,\mathbb{R})$, we consider $\mathbb{R}^5$ as having the structure of $\odot^2_\circ\mathbb{R}^3$. We can have the coordinate (8.1) on $\mathbb{R}^5$, with x,y,z,u,v being real valued. One then obtains the following non-holomorphic twistor correspondence:

$$\begin{tikzcd} & \mathbb{R}^5 \times \mathbb{P}(\mathbb{C}^3_\circ) \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ T & & \mathbb{R}^5. \end{tikzcd}$$

Definition 8.3 *The group $ESO(3,\mathbb{R})$ is defined by*

$$ESO(3,\mathbb{R}) = \{(A,B) \mid A \in SO(3,\mathbb{R}), B \in \mathbb{R}^5\},$$

with group composition rule (8.3).

Similar to that of the holomorphic case, the group $ESO(3,\mathbb{R})$ acts on T, $F = \mathbb{R}^5 \times \mathbb{P}(\mathbb{C}^3_\circ)$, $\mathbb{R}^5$ transitively and makes them homogeneous spaces of $ESO(3,\mathbb{R})$. We then have the following Lie groups and Lie algebras,

$$G^{\mathbb{C}} = ESO(3,\mathbb{C}) \qquad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{pmatrix}, \begin{pmatrix} a-ib+ie & b+e & c+d \\ b+e & a+ib-ie & ic-id \\ c+d & ic-id & -2a \end{pmatrix} \right\}$$

$$Q \text{ is as in (8.4)} \qquad \mathfrak{q} = \left\{ \begin{pmatrix} 0 & p & q \\ -p & 0 & qi \\ -q & -qi & 0 \end{pmatrix}, \begin{pmatrix} a-ib & b & c+d \\ b & a+ib & ic-id \\ c+d & ic-id & -2a \end{pmatrix} \right\}$$

$$L = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a_0 & 0 & f_0 \\ 0 & a_0 & g_0 \\ f_0 & g_0 & -2a_0 \end{pmatrix} \right\} \qquad \mathfrak{l} = \left\{ \begin{pmatrix} 0 & p & 0 \\ -p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & f \\ 0 & a & g \\ f & g & -2a \end{pmatrix} \right\}$$

$$K \cong SO(3,\mathbb{R}) \qquad \mathfrak{k} = \left\{ \left(\begin{pmatrix} 0 & p & q \\ -p & 0 & r \\ -q & -r & 0 \end{pmatrix}, 0 \right) \right\}$$

$$M = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0 \right\} \qquad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & p & 0 \\ -p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0 \right\},$$

where $\theta, a_0, f_0, g_0 \in \mathbb{R}$, and all other parameters are complex valued. Also one has

$$\mathfrak{r} = \mathfrak{k} \cap \mathfrak{q} = \left\{ \left(\begin{pmatrix} 0 & p & q \\ -p & 0 & qi \\ -q & -qi & 0 \end{pmatrix}, 0 \right) \right\}.$$

Lemma 8.4 *The adjoint representations of $G^{\mathbb{C}} = ESO(3, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{eso}(3, \mathbb{C})$ on $\mathfrak{eso}(3, \mathbb{C})$ are given by*

$$\begin{cases} Ad_{(A,B)}(X, Y) = (AXA^t, [B, AXA^t] + AYA^t) \\ ad_{(X,Y)}(X', Y') = ([X, X'], [X, Y'] + [Y, X']), \end{cases}$$

where $(A, B) \in ESO(3, \mathbb{C})$, $(X, Y), (X', Y') \in \mathfrak{eso}(3, \mathbb{C})$.

Proof: Straightforward calculations. □

We then have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with the K-module $\mathfrak{p}$ being $\mathfrak{p} \cong \mathbb{C}^5$, and

$$\mathfrak{q}/\mathfrak{r} = \mathfrak{p} \cap \mathfrak{q} = \left\{ \begin{pmatrix} a - ib & b & c + d \\ b & a + ib & ic - id \\ c + d & ic - id & -2a \end{pmatrix} \right\}.$$

Note L/M, the fibre of η, is contractible, therefore one can apply the non-holomorphic Penrose transform to this non-holomorphic twistor correspondence of $\mathbb{R}^5$.

Holomorphic line bundles on T

Definition 8.5 *The smooth vector bundle $E_{n,\lambda}$ on T is associated to the L-module given by the following representation of L*

$$\rho : \left(\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 & f \\ 0 & a & g \\ f & g & -2a \end{pmatrix} \right) \mapsto e^{-n\theta - \lambda a} \quad n \in \mathbb{Z}, \ \lambda \in \mathbb{C}.$$

Lemma 8.6 *The bundle $E_{n,\lambda}$ is a holomorphic, i.e. it can be associated to a $(\mathfrak{q}, L)$-module, if and only if $\lambda = 0$. Such bundles will be written as $\mathcal{O}\{n\}$ hereafter.*

Proof: When $\lambda = 0$, by

$$\left[\begin{pmatrix} 0 & p & q \\ -p & 0 & qi \\ -q & -qi & 0 \end{pmatrix}, \begin{pmatrix} 0 & u & v \\ -u & 0 & vi \\ -v & -vi & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & i(pv - qu) \\ 0 & 0 & qu - pv \\ -i(pv - qu) & -(qu - pv) & 0 \end{pmatrix},$$

one has that

$$\dot{\rho}(s) = -np, \qquad s \in \mathfrak{q},\ n \in \mathbb{C} \tag{8.5}$$

is a an representation of $\mathfrak{q}$. For it to agree with the differential of ρ when restricted to $\mathfrak{l}$, one then requires n to be integer valued. Thus $\mathcal{O}\{n\}$, $n \in Z$ is holomorphic.

Now in general consider, for $s, s' \in \mathfrak{q}$,

$$[s, s'] = \left(*, \begin{pmatrix} 2(pb' + qc' + qd') & 2i(pb' + qc') & i(pc' - pd') - 3qa' \\ 2i(pb' + qc') & -2(pb' + qc' - qd') & -(pc' - pd') - 3iqa' \\ i(pc' - pd') - 3qa' & -(pc' + pd') - 3a'qi & -4qd' \end{pmatrix} \right.$$

$$\left. - \begin{pmatrix} 2(p'b + q'c + q'd) & 2i(p'b + q'c) & i(p'c - p'd) - 3q'a \\ 2i(p'b + q'c) & -2(p'b + q'c - q'd) & -(p'c - p'd) - 3iq'a \\ i(p'c - p'd) - 3q'a & -(p'c + p'd) - 3iaq'i & -4q'd \end{pmatrix} \right).$$

Therefore $[\mathfrak{q}, \mathfrak{q}]$ is 5-dimensional. Then by the fact that

$$\{\text{ 1-dimensional representations of } \mathfrak{q}\} \cong (\mathfrak{q}/[\mathfrak{q}, \mathfrak{q}])^*,$$

the set of 1-dimensional representations of $\mathfrak{q}$ is only 1-dimensional ($\dim_{\mathbb{C}}(\mathfrak{q}) = 6$). As $\{n \in \mathbb{C}\}$ in (8.2) is already 1-dimensional, λ can only take on discrete values. However if $(E_{n,\lambda})_0$ is a $\mathfrak{q}$-module for a $\lambda \neq 0$, then $(E_{n,\lambda'})_0$ will be a $\mathfrak{q}$-module for an arbitrary λ' also. Therefore for $\lambda \neq 0$, the L-module cannot extend. □

Definition 8.7 *Let* $\mathbb{E}\{n\}$ *denote the bundle* $\eta^*\mathcal{O}\{n\}$ *on* F, $\mathbb{C}\{n\}$ *the corresponding* $(\mathfrak{r}, M)$*-module.*

Definition 8.8 *Let* $\mathcal{E}^\alpha$ *be the homogeneous vector bundle on* $\mathbb{R}^5$ *associated with the defining representation of* $SO(3)$, $\mathcal{E}_\alpha$ *its dual and* $\epsilon_{\alpha\beta}$ *the symmetric bilinear form on* $\mathcal{E}^\alpha$ *associated with the metric* g_{ab} *of* $\mathbb{R}^5$

Lemma 8.9 *Direct images* $\tau_*^p \mathbb{E}\{n\}$ *are*

$$\begin{cases} \tau_* \mathbb{E}\{n\} & \cong \ \odot_\circ^n \mathcal{E}^\alpha, \quad n \geq 0 \\ \tau_*^1 \mathbb{E}\{-n-1\} & \cong \ \odot_\circ^n \mathcal{E}^\alpha, \quad n \geq 0 \\ \textit{others vanish,} & \end{cases}$$

where $\odot_\circ^n \mathcal{E}^\alpha$ *is the totally tracefree part of* $\odot^n \mathcal{E}^\alpha$ *and we do not distinguish* $\mathcal{E}^\alpha$ *and* $\mathcal{E}_\alpha$.

Lemma 8.10 *One has the following short exact sequence of $(\mathfrak{r}, M)$-modules*

$$0 \longrightarrow \mathbb{C}\{-1\} \longrightarrow (\mathfrak{q}/\mathfrak{r})^* \longrightarrow \mathbb{C}^\alpha\{1\} \longrightarrow 0. \tag{8.6}$$

Proof: For simplicity, we think of $(\mathfrak{q}/\mathfrak{r})^*$ as a R-module first. For $r = \exp(r_0) \in R$, where $r_0 \in \mathfrak{r}$, one has that $\mathrm{Ad}^*(r)$ acts on $\begin{pmatrix} a^* - ib^* & b^* & c^* + d^* \\ b^* & a^* + ib^* & ic^* - id^* \\ c^* + d^* & ic^* - id^* & -2a^* \end{pmatrix} \in (\mathfrak{q}/\mathfrak{r})^*$ by

$$\begin{aligned}
a^* &\mapsto e^{0ip}[a^* - 3ic^*(q/p)y + 3ib^*(q/p)^2y^2] \\
b^* &\mapsto e^{-2ip}(b^*) \\
c^* &\mapsto e^{-ip}[c^* - 2b^*(q/p)y] \\
d^* &\mapsto e^{ip}[d^* + 2i(q/p)a^*y + 3c^*(q/p)^2y^2 - 2(q/p)^3b^*y^3],
\end{aligned}$$

where $y = (1 - e^{-ip})$.

This action can be written as

$$\begin{pmatrix} a^* + 3ib^* \\ ia^* + 3b^* \\ 3c^* \\ d^* \end{pmatrix} \longmapsto \begin{pmatrix} & & & 0 \\ & e^{-ip}r & & 0 \\ & & & 0 \\ e^{ip}* & e^{ip}* & e^{ip}* & e^{ip} \end{pmatrix} \begin{pmatrix} a^* + 3ib^* \\ ia^* + 3b^* \\ 3c^* \\ d^* \end{pmatrix},$$

where the action of r on the first three components is just the standard representation of r.

Now regard $(\mathfrak{q}/\mathfrak{r})^*$ as a $(\mathfrak{r}, M)$-module. We then obtain a short exact sequence of $(\mathfrak{r}, M)$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}\{-1\} & \longrightarrow & (\mathfrak{q}/\mathfrak{r})^* & \longrightarrow & \mathbb{C}^\alpha\{1\} & \longrightarrow & 0 \\
 & & \Cup & & \Cup & & \Cup & & \\
 & & d' & \longmapsto & \begin{pmatrix} 0 \\ 0 \\ 0 \\ d' \end{pmatrix} & & & & \\
 & & & & \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} & \longmapsto & \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}. & &
\end{array}$$

□

Lemma 8.11 *The direct images are as follows*

<table>
<tr><td></td><td>$\tau_*\mathcal{E}_\eta^{1,0}\{n\}$</td><td>$\tau_*\mathcal{E}_\eta^{2,0}\{n\}$</td><td>$\tau_*\mathcal{E}_\eta^{3,0}\{n\}$</td><td>$\tau_*\mathcal{E}_\eta^{4,0}\{n\}$</td></tr>
<tr><td>$n \geq 1$</td><td>$\mathcal{E}^\alpha \otimes \odot_\circ^{n+1}\mathcal{E}^\alpha$
$\oplus \odot_\circ^{n-1}\mathcal{E}^\alpha$</td><td rowspan="2">$\mathcal{E}^\alpha \otimes \odot_\circ^{n}\mathcal{E}^\alpha$
$\oplus$
$\mathcal{E}^\alpha \otimes \odot_\circ^{n+2}\mathcal{E}^\alpha$</td><td rowspan="3">$\mathcal{E}^\alpha \otimes \odot_\circ^{n+1}\mathcal{E}^\alpha$
$\oplus$
$\odot_\circ^{n+3}\mathcal{E}^\alpha$</td><td rowspan="4">$\odot_\circ^{n+2}\mathcal{E}^\alpha$</td></tr>
<tr><td>$n = 0$</td><td>$\mathcal{E}_\circ^{\alpha\beta}$</td></tr>
<tr><td>$n = -1$</td><td rowspan="3">0</td><td>$\odot^2\mathcal{E}^\alpha$</td></tr>
<tr><td>$n = -2$</td><td rowspan="2">0</td><td rowspan="2">0</td></tr>
<tr><td>$n \leq -3$</td><td>0</td></tr>
</table>

<table>
<tr><td></td><td>$\tau_*^1\mathcal{E}_\eta^{1,0}\{-m-1\}$</td><td>$\tau_*^1\mathcal{E}_\eta^{2,0}\{-m-1\}$</td><td>$\tau_*^1\mathcal{E}_\eta^{3,0}\{-m-1\}$</td><td>$\tau_*^1\mathcal{E}_\eta^{4,0}\{-m-1\}$</td></tr>
<tr><td>$m \leq 0$</td><td>0</td><td>0</td><td rowspan="2">0</td><td rowspan="2">0</td></tr>
<tr><td>$m = 1$</td><td rowspan="3">$\odot_\circ^{m+1}\mathcal{E}^\alpha$
$\oplus$
$\mathcal{E}^\alpha \otimes \mathcal{E}_\circ^{m-1}\mathcal{E}^\alpha$</td><td>$\odot^2\mathcal{E}^\alpha$</td></tr>
<tr><td>$m = 2$</td><td rowspan="2">$\mathcal{E}^\alpha \otimes \odot_\circ^{m}\mathcal{E}^\alpha$
$\oplus$
$\mathcal{E}^\alpha \otimes \odot_\circ^{m-2}\mathcal{E}^\alpha$</td><td>$\mathcal{E}_\circ^{\alpha\beta}$</td><td rowspan="2">$\odot_\circ^{m-2}\mathcal{E}^\alpha$</td></tr>
<tr><td>$m \geq 3$</td><td>$\mathcal{E}^\alpha \otimes \odot_\circ^{m-1}\mathcal{E}^\alpha$
$\oplus \odot_\circ^{m-3}\mathcal{E}^\alpha$</td></tr>
</table>

where $\tau_*^q\mathcal{E}_\eta^{1,0}\{n\}$ *stands for* $\tau_*^q(\mathbb{E}(\mathcal{E}_\eta^{1,0}\{n\}))$ *and we do not distinguish between* $\mathcal{E}_\alpha$ *and* $\mathcal{E}^\alpha$.

Proof: Tensoring the short exact sequence of vector bundles associated with (8.6) with $\mathbb{E}\{n\}$, we get

$$0 \longrightarrow \mathbb{E}\{n-1\} \longrightarrow \mathbb{E}(\mathcal{E}_\eta^{1,0}\{n\}) \longrightarrow \mathbb{E}^\alpha\{n+1\} \longrightarrow 0. \tag{8.7}$$

Taking direct images, we obtain

$$0 \longrightarrow \tau_*\mathbb{E}\{n-1\} \longrightarrow \tau_*(\mathbb{E}(\mathcal{E}_\eta^{1,0}\{n\})) \longrightarrow \tau_*\mathbb{E}^\alpha\{n+1\}$$
$$\longrightarrow \tau_*^1\mathbb{E}\{n-1\} \longrightarrow \tau_*^1(\mathbb{E}(\mathcal{E}_\eta^{1,0}\{n\})) \longrightarrow \tau_*^1\mathbb{E}^\alpha\{n+1\} \longrightarrow 0.$$

Note the s.e.s (8.7) doesn't split naturally on F. However, on $\mathbb{R}^5$, as the isotropy group at $\underline{0}$ is semi-simple, all short exact sequences split naturally. This enables us to work out direct images $\tau_*^p\mathbb{E}(\mathcal{E}_\eta^{1,0}\{n\})$, when the long exact sequence is reduced to some short exact sequence, using known results about $\tau_*^p\mathbb{E}\{n\}$.

To work out $\tau_*^p \mathbb{E}(\mathcal{E}_\eta^{2,0}\{n\})$ and $\tau_*^p \mathbb{E}(\mathcal{E}_\eta^{3,0}\{n\})$, instead of using (8.6), we start with short exact sequences

$$0 \longrightarrow \mathbb{C}^\alpha \longrightarrow \wedge^2(\mathfrak{q}/\mathfrak{r})^* \longrightarrow \mathbb{C}^\alpha\{2\} \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{C}^\alpha\{1\} \longrightarrow \wedge^3(\mathfrak{q}/\mathfrak{r})^* \longrightarrow \mathbb{C}\{3\} \longrightarrow 0,$$

respectively. Also we have $\mathbb{E}(\mathcal{E}_\eta^{4,0}) \cong \mathbb{E}\{2\}$.

After some straight forward calculations, we obtain the two tables. □

Corollary 8.12 *One has the isomorphisms:*

$$\tau_*\mathbb{E}(\mathcal{E}_\eta^{1,0}\{n\}) \cong \tau_*^1\mathbb{E}(\mathcal{E}_\eta^{1,0}\{-n-2\}), \qquad \tau_*\mathbb{E}(\mathcal{E}_\eta^{2,0}\{n\}) \cong \tau_*^1\mathbb{E}(\mathcal{E}_\eta^{2,0}\{-n-3\}),$$
$$\tau_*\mathbb{E}(\mathcal{E}_\eta^{3,0}\{n\}) \cong \tau_*^1\mathbb{E}(\mathcal{E}_\eta^{3,0}\{-n-4\}), \qquad \tau_*\mathbb{E}(\mathcal{E}_\eta^{4,0}\{n\}) \cong \tau_*^1\mathbb{E}(\mathcal{E}_\eta^{4,0}\{-n-5\}).$$

8.3 The Transform

In this section we compute $H^1(T, \mathcal{O}\{n\})$ in five cases (i) $n = 0$, (ii) $n = -2$, (iii) $n \geq 1$, (iv) $n \leq -3$, (v) $n = -1$, using the spectral sequence (5.27).

Proposition 8.13 *One has the isomorphism*

$$H^1(T, \mathcal{O}) \;\cong\; \frac{\left\{\nabla_{\alpha[\lambda}\psi^\alpha{}_{\rho]} = 0 \;\&\; \nabla^{[\alpha}{}_{(\beta}\psi^{\gamma]}{}_{\delta)} - \frac{1}{3}\epsilon_{\beta\delta}\nabla^{[\alpha}{}_\lambda\psi^{\gamma]\lambda} = 0\right\}}{\{\nabla_{\gamma\delta}\phi\}},$$

where $\phi \in \Gamma(\mathbb{R}^5, \mathcal{E})$, $\psi^{\alpha\beta} \in \Gamma(\mathbb{R}^5, \mathcal{E}_\circ^{\alpha\beta})$, $\nabla_{\alpha\beta}$ is the connection on $\mathbb{R}^5$, and $\epsilon_{\alpha\beta\gamma} = \epsilon_{[\alpha\beta\gamma]}$ is the volume form on $\mathbb{C}^3$.

Proof: We have, at the E_1 level of (5.27),

$$\begin{array}{|ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ \mathcal{E} & \xrightarrow{\beta} & \mathcal{E}_\circ^{\alpha\beta} & \xrightarrow{\alpha} & \mathcal{E}^\alpha \oplus \mathcal{E}^\alpha \otimes \odot_\circ^2\mathcal{E}_\alpha & \longrightarrow & \dots, \\ \hline \end{array}$$

where

$$\begin{array}{rccc} \beta: & \mathcal{E} & \longrightarrow & \mathcal{E}_\circ^{\alpha\beta} \\ & \Downarrow & & \Downarrow \\ & \phi & \longmapsto & \nabla^{\alpha\beta}\phi, \end{array}$$

$$\begin{array}{rccc} \alpha: & \mathcal{E}_\circ^{\alpha\beta} & \longrightarrow & \mathcal{E}^\alpha \oplus \mathcal{E}^\alpha \otimes \odot_\circ^2 \mathcal{E}_\alpha \\ & \Downarrow & & \Downarrow \\ & \psi^{\alpha\beta} & \longmapsto & \left(\epsilon^{\lambda\rho\kappa}\nabla_{\alpha\lambda}\psi^\alpha{}_\rho,\ \epsilon^{\alpha\gamma\kappa}(\nabla_{\alpha(\beta}\psi_{|\gamma|\delta)} - \frac{1}{3}\epsilon_{\beta\delta}\nabla_{\alpha\lambda}\psi_\gamma{}^\lambda)\right). \end{array}$$

As the spectral sequence converges at E_2, we have the result. □

Proposition 8.14 *One has the isomorphism*

$$H^1(T, \mathcal{O}\{-2\}) \;\cong\; \left\{\epsilon_{(\delta}{}^{\beta\gamma}\nabla_{\alpha)\beta}\phi_\gamma = 0 \;\&\; \nabla_{\beta\gamma}\phi^\gamma = 0\right\},$$

where $\gamma_\alpha \in \Gamma(\mathbb{R}^5, \mathcal{E}_\alpha)$.

Proof: At E_1 level, we have

$$\begin{array}{|ccccccc} \mathcal{E}_\alpha & \xrightarrow{\alpha} & \odot_\circ^2\mathcal{E}_\alpha \oplus \mathcal{E}_\alpha & \longrightarrow & \odot^2\mathcal{E}_\alpha & \longrightarrow & \cdots \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots, \\ \hline \end{array}$$

where

$$\begin{array}{rccc} \alpha: & \mathcal{E}_\alpha & \longrightarrow & \odot_\circ^2\mathcal{E}_\alpha \oplus \mathcal{E}_\alpha \\ & \Downarrow & & \Downarrow \\ & \phi_\alpha & \longmapsto & \left(\epsilon_{(\delta}{}^{\beta\gamma}\nabla_{\alpha)\beta}\phi_\gamma,\ \nabla_{\beta\gamma}\phi^\gamma\right). \end{array}$$

The spectral sequence converges at E_2, and thus gives us the result. □

Proposition 8.15 *For $n \geq 1$, one has the isomorphism*

$$H^1(T, \mathcal{O}\{n\}) \;\cong\; \frac{\left\{\begin{array}{l} \nabla_{\alpha(\gamma}\psi_{\lambda\ldots\rho)} - \frac{n-1}{2n-1}\epsilon_{(\gamma\lambda}\nabla_{|\alpha\sigma|}\psi^\sigma{}_{\ldots\rho)} + \epsilon_\alpha{}^{\eta\nu}\nabla_\eta{}^\mu\zeta_{\mu\nu\ldots\rho} = 0 \\ \&\ \nabla^{[\alpha}{}_{(\beta}\zeta^{\mu]}{}_{\nu\ldots\rho)} - \frac{n+1}{2n+3}\epsilon_{(\beta\nu}\nabla^{[\alpha}{}_{|\kappa|}\zeta^{\mu]\kappa}{}_{\ldots\rho)} = 0 \end{array}\right\}}{\left\{\epsilon_{(\lambda}{}^{\xi\eta}\nabla_{|\xi\kappa}\phi^\kappa_{\eta|\ldots\rho)}\right\} \oplus \left\{\nabla_{\mu(\nu}\phi_{\gamma\ldots\rho)} - \frac{n}{2n+1}\epsilon_{(\nu\gamma}\nabla_{|\mu\sigma|}\phi^\sigma{}_{\ldots\rho)}\right\}},$$

where $\phi_{\gamma\ldots\rho} \in \Gamma(\mathbb{R}^5, \odot_\circ^n\mathcal{E}_\alpha)$, $\psi_{\lambda\ldots\rho} \in \Gamma(\mathbb{R}^5, \odot_\circ^{n-1}\mathcal{E}_\alpha)$ *and* $\zeta_{\mu\nu\ldots\rho} \in \Gamma(\mathbb{R}^5, \mathcal{E}_\alpha \otimes \odot_\circ^{n+1}\mathcal{E}_\alpha)$.

Proof: At E_1 level, we have

$$
\left|\begin{array}{ccccccc}
0 & \to & 0 & \to & 0 & \to \\
\odot_\circ^n \mathcal{E}_\alpha & \stackrel{\beta}{\to} & \odot_\circ^{n-1}\mathcal{E}_\alpha \oplus \mathcal{E}_\alpha \otimes \odot_\circ^{n+1}\mathcal{E}_\alpha & \stackrel{\alpha}{\to} & \mathcal{E}_\alpha \otimes \odot_\circ^{n}\mathcal{E}_\alpha \oplus \mathcal{E}_\alpha \otimes \odot_\circ^{n+2}\mathcal{E}_\alpha & \to,
\end{array}\right.
$$

where

$$
\beta : \qquad \phi_{\gamma\dots\rho} \longmapsto \left(\epsilon_{(\lambda}{}^{\xi\eta}\nabla_{|\xi\kappa}\phi^{\kappa}_{\eta|\dots\rho)},\ \nabla_{\mu(\nu}\phi_{\gamma\dots\rho)} - \tfrac{n}{2n+1}\epsilon_{(\nu\gamma}\nabla_{|\mu\sigma|}\phi^{\sigma}{}_{\dots\rho)}\right)
$$

$$
\begin{aligned}
\alpha : \ (\psi_{\lambda\dots\rho},\ \zeta_{\mu\nu\dots\rho}) \longmapsto \Big(&\nabla_{\alpha(\gamma}\psi_{\lambda\dots\rho)} - \tfrac{n-1}{2n-1}\epsilon_{(\gamma\lambda}\nabla_{|\alpha\sigma|}\psi^{\sigma}{}_{\dots\rho)} + \epsilon_{\alpha}{}^{\eta\nu}\nabla_{\eta}{}^{\mu}\zeta_{\mu\nu\dots\rho}, \\
&\epsilon_{\delta}{}^{\alpha\mu}\nabla_{\alpha(\beta}\zeta_{|\mu|\nu\dots\rho)} - \tfrac{n+1}{2n+3}\epsilon_{(\beta\nu}\epsilon_{|\delta}{}^{\alpha\mu}\nabla_{\alpha\kappa}\zeta_{\mu|}{}^{\kappa}{}_{\dots\rho)}\Big).
\end{aligned}
$$

The spectral sequence converges at E_2 and gives the result. □

Proposition 8.16 *For $n = -m - 1 \leq -3$, one has*

$$
H^1(T, \mathcal{O}\{-m-1\}) \cong \left\{\begin{array}{l}
\epsilon_{(\sigma}{}^{\alpha\gamma}\nabla_{|\alpha|\beta}\phi_{|\gamma|\delta\dots\rho)} - \tfrac{m-1}{2m+1}\epsilon_{(\beta\delta}\epsilon_{\sigma}{}^{\alpha\gamma}\nabla_{|\alpha\kappa}\phi^{\kappa}_{\gamma|\dots\rho)} = 0 \\
\&\nabla^{\beta}_{\alpha}\phi_{\beta\dots\rho} = 0
\end{array}\right\},
$$

where $\phi_{\gamma\delta\dots\rho} \in \Gamma(\mathbb{R}^5, \odot^m\mathcal{E}_\alpha)$.

Proof: At E_1 level, we have

$$
\left|\begin{array}{ccccccc}
\odot^m\mathcal{E}_\alpha & \xrightarrow{\alpha} & \odot_\circ^{m+1}\mathcal{E}_\alpha \oplus \mathcal{E}_\alpha \otimes \odot_\circ^{m-1}\mathcal{E}_\alpha & \longrightarrow & * & \longrightarrow & \cdots \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots,
\end{array}\right.
$$

where

$$
\alpha : \ \phi_{\gamma\delta\dots\rho} \longmapsto \left(\epsilon_{(\sigma}{}^{\alpha\gamma}\nabla_{|\alpha|\beta}\phi_{|\gamma|\delta\dots\rho)} - \tfrac{m-1}{2m+1}\epsilon_{(\beta\delta}\epsilon_{\sigma}{}^{\alpha\gamma}\nabla_{|\alpha\kappa}\phi^{\kappa}_{\gamma|\dots\rho)},\ \nabla^{\beta}_{\alpha}\phi_{\beta\dots\rho}\right).
$$

The spectral sequence converges at E_2 and gives the result. □

Proposition 8.17 *One has the isomorphism*

$$
H^1(T, \mathcal{O}\{-1\}) \cong \{\nabla_{(\alpha}{}^{\beta}\nabla_{\gamma)\beta}f = 0\}, \quad f \in \Gamma(\mathbb{R}^5, \mathcal{E}).
$$

Proof: At E_1 level, we have

$$
\left|\begin{array}{ccccccccc}
\mathcal{E} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
0 & \longrightarrow & 0 & \longrightarrow & \odot^2\mathcal{E}^\alpha & \xrightarrow{\alpha} & \mathcal{E}^\alpha \oplus \odot_\circ^2\mathcal{E}^\alpha & \longrightarrow & \dots,
\end{array}\right.
$$

which gives rise to the following at E_2 level:

$$
\begin{tikzcd}
\Gamma(\mathbb{R}^5,\mathcal{E}) \arrow[drr, "D"] & 0 & 0 & \cdots \\
0 & 0 & \Gamma(\mathbb{R}^5, ker(\alpha)) & \cdots
\end{tikzcd}
$$

Our main task is to identify the operator D.

First, we introduce S_0, a submodule of $(\mathfrak{q}/\mathfrak{r})^*$ by the following short exact sequence of $(\mathfrak{r}, M)$-modules

$$
\begin{array}{ccccc}
0 \longrightarrow \mathbb{C}\{-1\} & \xrightarrow{\mu} & S_0 & \xrightarrow{\nu} & \mathbb{C} \longrightarrow 0 \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
d^* & \longmapsto & \binom{0}{d^*} & & \\
 & & \binom{a^*}{d^*} & \longmapsto & a^*,
\end{array}
$$

recalling the definition of a^*, d^* in $(\mathfrak{q}/\mathfrak{r})^*$. This induces a short exact sequence of vector bundles:

$$
0 \longrightarrow \mathbb{C}\{-1\} \xrightarrow{\mu} S \xrightarrow{\nu} \mathbb{C} \longrightarrow 0, \tag{8.8}
$$

where we use S to stand for the vector bundle associated to S_0. Next, we introduce some abbreviation:

$$
t_1 := \left(0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right) \qquad t_2 := \left(0, \begin{pmatrix} -i & 1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)
$$

$$
t_3 := \left(0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}\right) \qquad t_4 := \left(0, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}\right)
$$

$$
t_5 := \left(0, \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \qquad a := \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}, 0\right)
$$

$$
m := \left(\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0\right).
$$

Note $t_1, t_2, t_3, t_4 \in \mathfrak{q}/\mathfrak{r}$, $a \in \mathfrak{r}$, while $t_5 \in \mathfrak{p}$ but not in $\mathfrak{q}/\mathfrak{r} = \mathfrak{p} \cap \mathfrak{q}$. We have

$$
\begin{array}{lll}
\mathrm{Ad}_m t_1 = t_1, & \mathrm{Ad}_m t_2 = e^{2i\theta} t_2, & \mathrm{Ad}_m t_3 = e^{i\theta} t_3, \\
\mathrm{Ad}_m t_4 = e^{-i\theta} t_4, & \mathrm{Ad}_m t_5 = e^{-2i\theta} t_5; & \\
[a, t_1] = -3t_3, & [a, t_2] = 0, & [a, t_3] = 2it_2, \\
[a, t_4] = 2t_1, & [a, t_5] = -2it_4. &
\end{array}
$$

Now, consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \text{Œ}\{-1\} & \xrightarrow{\mu} & \text{Œ}(S) & \xrightarrow{\nu} & \text{Œ} & \to & 0 \\ & & \wr\| & & \downarrow\nabla' & & \downarrow\Box' & & \\ & \to & \text{Œ}\{-1\} & \xrightarrow{\partial_1} & \text{Œ}(\mathcal{E}_\eta^{1,0}\{-1\}) & \xrightarrow{\partial_2} & \odot^2\text{Œ}^\alpha & \to & \cdots \end{array}$$

where the first row is just (8.8) and the second row is the $\mathcal{E}_\eta^{p,0}\{-1\}$ resolution. Note that S is a subbundle of $\mathcal{E}_\eta^{1,0}$ rather than of $\mathcal{E}_\eta^{1,0}\{-1\}$.

In the first square, μ is defined by

$$\langle t_1, \mu s\rangle = 0, \quad \langle t_4, \mu s\rangle = s,$$

where s is a section of $\text{Œ}\{-1\}$. The operator ∂_1 is defined by

$$\langle X, \partial_1 s\rangle = Xs + \dot\rho(X)s, \quad X \in \mathfrak{p}\cap\mathfrak{q}.$$

Since $\dot\rho(t_i) = 0$ for $i = 1, 2, 3, 4$, we just get

$$\langle t_i, \partial_1 s\rangle = t_i s, \quad i = 1, 2, 3, 4.$$

To say ω is a smooth section of $\mathcal{E}_\eta^{1,0}\{n\}$ is to say

$$\begin{aligned} \langle t_1, \omega(gm)\rangle &= e^{in\theta}\langle \mathrm{Ad}_m t_1, \omega(g)\rangle &= e^{in\theta}\langle t_1, \omega(g)\rangle \\ \langle t_2, \omega(gm)\rangle &= e^{in\theta}\langle \mathrm{Ad}_m t_2, \omega(g)\rangle &= e^{i(n+2)\theta}\langle t_2, \omega(g)\rangle \\ \langle t_3, \omega(gm)\rangle &= e^{in\theta}\langle \mathrm{Ad}_m t_3, \omega(g)\rangle &= e^{i(n+1)\theta}\langle t_3, \omega(g)\rangle \\ \langle t_4, \omega(gm)\rangle &= e^{in\theta}\langle \mathrm{Ad}_m t_4, \omega(g)\rangle &= e^{i(n-1)\theta}\langle t_4, \omega(g)\rangle. \end{aligned}$$

For ω to be compatible with the involutive structure one needs in addition that

$$\begin{aligned} a\langle t_1, \omega\rangle &= \langle [a, t_1], \omega\rangle &= -3\langle t_3, \omega\rangle \\ a\langle t_2, \omega\rangle &= \langle [a, t_2], \omega\rangle &= 0 \\ a\langle t_3, \omega\rangle &= \langle [a, t_3], \omega\rangle &= 2i\langle t_2, \omega\rangle \\ a\langle t_4, \omega\rangle &= \langle [a, t_4], \omega\rangle &= 2\langle t_1, \omega\rangle. \end{aligned}$$

In particular, if ω is a section of $\mathcal{E}_\eta^{1,0}\{-1\}$ compatible with the involutive structure, we just let $n = -1$. If ω is a section of S, then we have $n = 0$ and the middle two equations of each set of equations are dismissed.

By the requirement that if ω is an involutive section of S, then $\nabla'\omega$ is an involutive section of $\mathcal{E}^{1,0}_{\eta}\{-1\}$, and the fact that the first square commutes, we can work out what $\nabla'\omega$ is. It is characterized by

$$\begin{aligned}
\langle t_1, \nabla'\omega\rangle &= -3t_4\langle t_1,\omega\rangle + t_1\langle t_4,\omega\rangle\\
\langle t_2, \nabla'\omega\rangle &= it_3\langle t_1,\omega\rangle + t_2\langle t_4,\omega\rangle\\
\langle t_3, \nabla'\omega\rangle &= \tfrac{4}{3}t_1\langle t_1,\omega\rangle + t_3\langle t_4,\omega\rangle\\
\langle t_4, \nabla'\omega\rangle &= -4it_5\langle t_1,\omega\rangle + t_4\langle t_4,\omega\rangle.
\end{aligned}$$

Next consider the second square. The map ν is defined by

$$\omega \mapsto \langle t_1, \omega\rangle,$$

where ω is a section of $\mathbb{C}\!\mathrm{E}(S)$, while ∂_2 is characterized by

$$\begin{aligned}
\langle t_i, t_j, \partial_2\omega\rangle &= t_i\langle t_j,\omega\rangle - t_j\langle t_i,\omega\rangle - \langle [t_i,t_j],\omega\rangle\\
&\quad +\dot\rho(t_i)\langle t_j,\omega\rangle - \dot\rho(t_j)\langle t_i,\omega\rangle\\
&= t_i\langle t_j,\omega\rangle - t_j\langle t_i,\omega\rangle,
\end{aligned}$$

where we have used $[t_i,t_j]=0$ and $\dot\rho(t_i)=0$ for $i,j=1,2,3,4$, $\omega\in\mathbb{C}\!\mathrm{E}(\mathcal{E}^{1,0}_{\eta}\{-1\})$.

As $\Box'\nu(\omega) = \partial_2\nabla'\omega$ for ω a section of S, we can identify $\Box'$ after working out $\partial_2\nabla'$. It is a straightforward calculation to compute $\partial_2\nabla'$. We just list the results here

$$\begin{aligned}
\langle t_1,t_2,\Box'\nu(\omega)\rangle &= \langle t_1,t_2,\partial_2\nabla'\omega\rangle &&= (it_1t_3+3t_2t_4)\nu(\omega),\\
\langle t_1,t_3,\Box'\nu(\omega)\rangle &= \langle t_1,t_3,\partial_2\nabla'\omega\rangle &&= (\tfrac{4}{3}t_1t_1+3t_3t_4)\nu(\omega),\\
\langle t_1,t_4,\Box'\nu(\omega)\rangle &= \langle t_1,t_4,\partial_2\nabla'\omega\rangle &&= (-4it_1t_5+3t_4t_4)\nu(\omega),\\
\langle t_2,t_3,\Box'\nu(\omega)\rangle &= \langle t_2,t_3,\partial_2\nabla'\omega\rangle &&= (\tfrac{4}{3}t_2t_1-it_3t_3)\nu(\omega),\\
\langle t_2,t_4,\Box'\nu(\omega)\rangle &= \langle t_2,t_4,\partial_2\nabla'\omega\rangle &&= (-4it_2t_5-it_4t_3)\nu(\omega),\\
\langle t_3,t_4,\Box'\nu(\omega)\rangle &= \langle t_3,t_4,\partial_2\nabla'\omega\rangle &&= (-4it_3t_5-\tfrac{4}{3}t_4t_1)\nu(\omega).
\end{aligned}$$

Writing $\nabla_\alpha{}^\beta$ explicitly as a matrix, we have, using (8.1),

$$\nabla_\alpha{}^\beta \propto \begin{pmatrix} \frac{1}{\sqrt3}\partial_x+\partial_z & \partial_y & \partial_u\\ \partial_y & \frac{1}{\sqrt3}\partial_x-\partial_z & \partial_v\\ \partial_u & \partial_v & \frac{-2}{\sqrt3}\partial_x \end{pmatrix},$$

where $\partial_x := \frac{\partial}{\partial x}$, $\partial_y := \frac{\partial}{\partial y}$ etc. By the relationship between two coordinate systems (8.1) and (8.2), one obtains that

$$\nabla_\alpha{}^\beta \propto \begin{pmatrix} \frac{1}{3}\partial_1 + \frac{i}{2}\partial_2 - \frac{i}{2}\partial_5 & \frac{1}{2}\partial_2 + \frac{1}{2}\partial_5 & \frac{1}{2}\partial_3 + \frac{1}{2}\partial_4 \\ \frac{1}{2}\partial_2 + \frac{1}{2}\partial_5 & \frac{1}{3}\partial_1 - \frac{i}{2}\partial_2 + \frac{i}{2}\partial_5 & -\frac{i}{2}\partial_3 + \frac{i}{2}\partial_4 \\ \frac{1}{2}\partial_3 + \frac{1}{2}\partial_4 & -\frac{i}{2}\partial_3 + \frac{i}{2}\partial_4 & -\frac{2}{3}\partial_1 \end{pmatrix},$$

where $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1 \ldots 5$.

By matrix multiplication, we can write $\nabla_\alpha{}^\beta \nabla_\beta{}^\gamma$ explicitly as a matrix also. Note it is automatically symmetric in α, γ, but not tracefree. The kernel of $\nabla_\alpha{}^\beta \nabla_\beta{}^\gamma$ is composed of functions $f \in \Gamma(\mathbb{R}^5, \mathcal{E})$ satisfying, after some arrangement,

$$\begin{aligned} (i\partial_1\partial_3 + 3\partial_2\partial_4)f &= (\tfrac{4}{3}\partial_1\partial_1 + 3\partial_3\partial_4)f = (-4i\partial_1\partial_5 + 3\partial_4\partial_4)f \\ = (\tfrac{4}{3}\partial_2\partial_1 - i\partial_3\partial_3)\nu(\omega) &= (-4i\partial_2\partial_5 - i\partial_4\partial_3)f = (-4i\partial_3\partial_5 - \tfrac{4}{3}\partial_4\partial_1)f = 0. \end{aligned}$$

Now compare this with the formulas that characterize $\Box'$. We see immediately that $\nabla_{(\alpha}{}^\beta \nabla_{\gamma)\beta}$ is proportional to the zeroth direct image map of $\Box'$. Arguments similar to that in the proof of Proposition 2.12 then give us $D \cong \nabla_{(\alpha}{}^\beta \nabla_{\gamma)\beta}$. We thus have the result. $\Box$

Appendix: An alternative approach

By considering the reduction of $SO(5,\mathbb{C})$ to $SL(2,\mathbb{C}) \hookrightarrow SO(5,\mathbb{C})$, $\mathbb{C}^5$ can be equipped with the structure of $\odot^4\mathbb{C}^2$, and one can write its holomorphic tangent bundle $\mathcal{O}^a$ as $\odot^4\mathcal{O}^A$, where $\mathcal{O}^A$ is a rank two vector bundle equipped with a skew form ϵ_{AB}. In this context, a vector is quadruple null if it is of the form $\alpha^A\alpha^B\alpha^C\alpha^D$. The twistor correspondence is

$$T \xleftarrow{\mu} \mathbb{C}^5 \times \mathbb{CP}_1 \xrightarrow{\nu} \mathbb{C}^5,$$

where ν is the obvious forgetful map and μ is given by

$$\mu(X^{ABCD}, [\pi_A]) = (\pi_A, X^{ABCD}\pi_A\pi_B\pi_C\pi_D)/\sim,$$

where $\sim$ is given by $(\pi_A, \xi) \sim (\lambda\pi_A, \lambda^4\xi)$, $\lambda \in \mathbb{C}^*$. The group $SL(2,\mathbb{C})\ltimes\mathbb{C}^5$ acts transitively on these three spaces. One can define homogeneous holomorphic line bundle $\mathcal{O}(n)$, $n \in \mathbb{Z}$, on T in the usual way (such that it agrees with the pull back of $\mathcal{O}_{\mathbb{P}_1}(n)$). The holomorphic Penrose transform then yields the following isomorphisms, which, when restricted to $\mathbb{R}^5$ and n is even, are alternative forms of propositions 8.13—8.17.

Proposition 8.18 *For $n \geq -1$,*

$$H^1(T, \mathcal{O}(n)) \;\cong\; \frac{\left\{\nabla^{PQR(H}\Psi_{PQR}{}^{A\ldots EG)} = 0 \;\&\; \nabla^{R}{}_{(MN}{}^{(H}\Psi_{PQ)R}{}^{A\ldots EG)} = 0\right\}}{\left\{\nabla^{(G}_{PQR}\phi^{A\ldots E)}\right\}},$$

where $\Psi_{PQR}{}^{A\ldots EG} \in \Gamma(\mathbb{C}^5, \mathcal{O}_{(PQR)} \otimes \odot^{n+1}\mathcal{O}^A)$, $\phi^{A\ldots E} \in \Gamma(\mathbb{C}^5, \odot^n\mathcal{O}^A)$ and $\odot^{-1}\mathcal{O}^A$ is taken to be vacuous.

Proposition 8.19 *For $n = -m - 2 \leq -3$,*

$$H^1(T, \mathcal{O}(n)) \;\cong\; \left\{\nabla_{PQR}{}^{A}\phi_{AB\ldots D} = 0\right\},$$

where $\phi_{AB\ldots D} \in \Gamma(\mathbb{C}^5, \odot^m\mathcal{O}_A)$.

Proposition 8.20

$$H^1(T, \mathcal{O}(-2)) \;\cong\; \left\{\Delta\phi = 0 \;\&\; \nabla^{LM}{}_{(PQ}\nabla_{EF)LM}\phi = 0\right\}, \quad \phi \in \Gamma(\mathbb{C}^5, \mathcal{O}).$$

Chapter 9

A Penrose Transform for M^5

In this chapter we investigate into the applicability of the Penrose transform techniques to non-compact symmetric spaces other than hyperbolic 3-space. The symmetric space $SL(3,\mathbb{R})/SO(3,\mathbb{R})$, which we shall write M^5 hereafter, is non-compact and has a natural twistor correspondence, see Sections 9.2. The non-holomorphic Penrose transform for this correspondence is discussed in Section 9.3. The complexified space $\mathbb{M}^5$ of M^5 has a natural holomorphic twistor correspondence also, see Section 9.1, which is in fact a 'flat' example of a more general correspondence discussed in Chapter 10. In an Appendix, some formulae used in the identification of direct image maps are derived.

9.1 Holomorphic twistor correspondence

Let $\mathbb{M}^5$ denote the set of non-degenerate conics in $\mathbb{CP}_2 = \mathbb{P}(\mathbb{C}^3)$. Identifying the space of conics with $\mathbb{P}(\odot^2\mathbb{C}^{3*}) \cong \mathbb{CP}_5$, we have

$$\mathbb{M}^5 = \mathbb{P}(\odot^2\mathbb{C}^{3*})\backslash W,$$

where W is the subvariety $\{X \mid \det X = 0\}$.

A point p in $\mathbb{CP}_2$ corresponds to all non-degenerate conics through p, which prescribe a hypersurface Σ_p in $\mathbb{M}^5$ isomorphic to $\mathbb{P}_4\backslash(\mathbb{P}_4 \cap W)$. We shall call such surfaces α-surfaces. See [13] for an explicit construction of Σ_p.

Let $\mathbb{F} = \{(p,q) \mid p \in \mathbb{P}_2,\ q \in \mathbb{M}^5, \text{ and } q \in \Sigma_p\}$, one then has the holomorphic twistor correspondence

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{CP}_2 & & \mathbb{M}^5, \end{tikzcd}$$

with the standard fibre of ν being $\mathbb{CP}_1$.

Differential geometric structures on $\mathbb{M}^5$

Proposition 9.1 *The manifold $\mathbb{M}^5$ has a natural projective structure induced from that of $\mathbb{P}_5$ and with respect to which all α-surfaces are totally geodesic.*

Let $V \subset W$ denote the image of the Veronese map $\mathbb{P}(\mathbb{C}^{3^*}) \longrightarrow \mathbb{P}(\odot^2\mathbb{C}^{3^*})$ given by

$$[Z] \longmapsto [Z \otimes Z],$$

so that V corresponds to conics of rank 1.

Definition 9.2 *As W is a cubic surface, a generic line in $\mathbb{P}_5$ intersects W at three points. A vector is said to be 3-null iff the line it generates intersects W at one triple point. A vector is is said to be 4-null if, in addition, the triple point actually lies in V.*

Proposition 9.3 *There exists a unique conformal structure on $\mathbb{M}^5$ such that with respect to which 3-null vectors are null.*

Proof: See Proposition 10.3, noting that Definition 9.2 agrees with the definition of 3-nullness therein. □

The manifold $\mathbb{M}^5$ in fact has a preferred complex holomorphic Riemannian metric, cf. [32],

$$ds^2 = \mathrm{Trace}((Y^{-1}dY)^2), \quad Y \in \mathbb{M}^5. \tag{9.1}$$

The geodesics through $I \in \mathbb{M}^5$ in the direction $X \in T\mathbb{M}^5_x$ is in particular given by e^{tX}.

Lemma 9.4 *Let X be a 3×3 symmetric tracefree matrix, then one has*

$$e^{tX} = aI + bX + cX^2, \quad \textit{for some } a, b, c.$$

Proof: Standard calculation. □

Corollary 9.5 *The geodesics of the metric connection is in general different from that of the projective structure.*

9.2 Non-holomorphic twistor correspondence

Let M^5 denote the space of non-degenerate *real* conics (i.e. invariant under complex conjugation) in $\mathbb{P}_2$. Then M^5 can be written as

$$M^5 = \mathbb{P}(\odot^2\mathbb{R}^{3*})\backslash\{X \mid \det X = 0\}.$$

We have a non-holomorphic correspondence

$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathbb{CP}_2\backslash\mathbb{RP}_2 & & M^5, \end{tikzcd}$$

where an element of F is a real conic in $\mathbb{CP}_2\backslash\mathbb{RP}_2$ with a point on it. The standard fibre of τ is $\mathbb{CP}_1$, while for $p \in \mathbb{CP}_2\backslash\mathbb{RP}_2$, $\tau(\eta^{-1}(p))$ is a real 3-dimensional submanifold of M^5. The manifold M^5 again has a natural Riemannian metric, cf. [32].

Symmetry group $SL(3,\mathbb{R})$

Consider the transitive group actions of $SL(3,\mathbb{R})$: for $g \in SL(3,\mathbb{R})$,

$$\begin{aligned}
&Z \longmapsto gZ && \text{on } \mathbb{CP}_2\backslash\mathbb{RP}_2,\\
&(Z,X)/\sim \longmapsto (gZ, g^{t^{-1}}Xg^{-1})/\sim && \text{on } F,\\
&X \longmapsto g^{t^{-1}}Xg^{-1} && \text{on } M^5.
\end{aligned}$$

Note: It is clear that $M^5 = SL(3,\mathbb{R})/SO(3,\mathbb{R})$. However $\mathbb{M}^5$ is not equal to $SL(3,\mathbb{C})/SO(3,\mathbb{C})$, the determinant one surface of the space of 3×3 symmetric complex matrices. Instead, $SL(3,\mathbb{C})/SO(3,\mathbb{C})$ is a $3:1$ covering of $\mathbb{M}^5$.

We now list various subgroups and subalgebras here.

$$L = \left\{\begin{pmatrix} a & b & c\\ -b & a & f\\ 0 & 0 & k\end{pmatrix} \in SL(3,\mathbb{R})\right\}, \quad M = \left\{\begin{pmatrix} \cos(\theta) & \sin(\theta) & 0\\ -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}\right\},$$

$$\mathfrak{q} = \left\{\begin{pmatrix} a & b & c\\ d & a+(b+d)i & f\\ g & gi & -2a-(b+d)i\end{pmatrix}\right\}, \quad \mathfrak{l} = \left\{\begin{pmatrix} a & b & c\\ -b & a & f\\ 0 & 0 & -2a\end{pmatrix}\right\},$$

$$\mathfrak{r} = \left\{\begin{pmatrix} 0 & b & g\\ -b & 0 & gi\\ -g & -gi & 0\end{pmatrix}\right\}, \quad \mathfrak{m} = \left\{\begin{pmatrix} 0 & b & 0\\ -b & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\},$$

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}, \text{ where } \mathfrak{p} = \left\{\begin{pmatrix} e & f & g\\ f & k & h\\ g & h & -e-k\end{pmatrix}\right\}.$$

Note, L/M, the fibre of η, is contractible.

Since $\mathfrak{l}$ is solvable and any irreducible representation of a solvable Lie algebra is one-dimensional, we only need to consider line bundles on G/L.

Lemma 9.6 *The representation of L*

$$\begin{pmatrix} a & b & c\\ -b & a & f\\ 0 & 0 & k\end{pmatrix} \overset{\rho}{\longmapsto} (a+bi)^{-n}k^{-\lambda} \quad n\in\mathbb{Z}, \lambda\in\mathbb{C},$$

cannot be extended to gives a $(\mathfrak{q},L)$-module, unless the continuous parameter λ is zero.

Proof: Use the same argument as that in the proof of Lemma 8.2 $\square$

Definition 9.7 *Define $\mathcal{O}\{n\}$ to be the line bundle associated to the $(\mathfrak{q}, L)$-module given by the representation of L as above, with $\lambda = 0$, and the representation $\dot{\rho}$ of $\mathfrak{q}$:*

$$\begin{pmatrix} a & b & c \\ d & a+(b+d)i & f \\ g & gi & -2a-(b+d)i \end{pmatrix} \xmapsto{\dot{\rho}} -n(a+bi), \qquad n \in \mathbb{Z}.$$

Note: The bundles $\mathcal{O}\{n\}$ can be extended to be bundles over $\mathbb{CP}_2 = SL(3,\mathbb{C})/Q$ also. The resulting line bundles $\mathcal{O}\{n\}$ on $\mathbb{CP}_2$ is in fact the n^{th} tensor product of the usual hyperplane section bundle on $\mathbb{CP}_2$.

Unlike the $\mathbb{R}^5$ case, we have $\mathfrak{q}/\mathfrak{r} \neq \mathfrak{p} \cap \mathfrak{q}$. Nevertheless, if we think of $\mathfrak{q}/\mathfrak{r}$ as the tangent space of Q/R at $I \in Q/R$, then one can write

$$(\mathfrak{q}/\mathfrak{r}) \cong \left\{ \begin{pmatrix} a-ib & b & c+d \\ b & a+ib & ic-id \\ c+d & ic-id & -2a \end{pmatrix} \right\}.$$

Therefore the structures of $\mathfrak{q}/\mathfrak{r}$ as an $(\mathfrak{r}, M)$-module is just like that of the $\mathbb{R}^5$ case. Then by similar computation as that for $\mathbb{R}^5$, we can obtain direct images, see Chapter 8, with $\mathbb{E}\{n\} = \eta^*\mathcal{O}\{n\}$ and $\mathcal{E}^\alpha$ is the bundle induced by the standard representation of $SO(3,\mathbb{R})$. The spectral sequence follows the same pattern also. The only difference is on the differential operators: $\widetilde{\Phi}$, as is characterized in (5.32), is in general not zero here. To actually work out direct image maps, it is better to use the method below.

Symmetry group $\widetilde{SL(3,\mathbb{R})}$

Let $\widetilde{SL(3,\mathbb{R})}$ be the universal covering group of $SL(3,\mathbb{R})$. It then acts on the twistor correspondence transitively by projecting to $SL(3,\mathbb{R})$ first. We then obtain $\mathbb{CP}_2 \backslash \mathbb{RP}_2 = \widetilde{SL(3,\mathbb{R})}/\widetilde{L}$ and $M^5 = \widetilde{SL(3,\mathbb{R})}/SU(2)$, where $\widetilde{L}$ is the preimage of L in $\widetilde{SL(3,\mathbb{R})}$.

Definition 9.8 *Let $\mathcal{O}(n)$ be the line bundle on $\mathbb{CP}_2\backslash\mathbb{RP}_2$ associated to the unique $(\mathfrak{q}, \widetilde{L})$-module determined by*

$$\begin{pmatrix} a & b & c \\ d & a+(b+d)i & f \\ g & gi & -2a-(b+d)i \end{pmatrix} \overset{\dot\rho}{\longmapsto} -\frac{n}{2}(a+bi), \quad n \in \mathbb{Z}.$$

It is clear that $\mathcal{O}\{n\} \cong \mathcal{O}(2n)$, $n \in \mathbb{Z}$. We shall use $\mathcal{E}(n)$ to denote $\eta^*\mathcal{O}(n)$ as usual.

Definition 9.9 *The bundle $\mathcal{E}^A$ is the homogeneous vector bundle on M^5 associated with the self representation of $SU(2)$.*

The bundle $\mathcal{E}^A$ is equipped with a spinorial connection ∇_{ABCD}, cf. Lemma 5.41, which, when acting on vectors, is the usual Levi-Civita connection. We also have a skewsymmetric form ϵ_{AB} associated with the metric.

Definition 9.10 *Define $\Box_{ABCDEF}$, $\Box_{AB}$ etc as follows,*

$$\begin{aligned} \nabla_a\nabla^b\mu^N &= \nabla_{ABCD}\nabla^{EFGH}\mu^N \\ &= \Box_{ABCD}{}^{EFGH}\mu^N + \epsilon_{(A}{}^{(E}\Box_{BCD)}{}^{FGH)}\mu^N + \epsilon_{(A}{}^{(E}\epsilon_B{}^F\Box_{CD)}{}^{GH)}\mu^N \\ &\quad + \epsilon_{(A}{}^{(E}\epsilon_B{}^F\epsilon_C{}^G\Box_{D)}{}^{H)}\mu^N + \epsilon_{(A}{}^{(E}\epsilon_B{}^F\epsilon_C{}^G\epsilon_{D)}{}^{H)}\Box\mu^N, \end{aligned}$$

where $\Box_{ABCDEF} = \Box_{(ABCDEF)}$, $\Box_{AB} = \Box_{(AB)}$ etc.

Lemma 9.11 *In terms of ∇_{ABCD}, they are*

$$\Box_{ABCDEFGH}\mu^N = \nabla_{(ABCD}\nabla_{EFGH)}\mu^N, \quad \Box_{ABCDEF}\mu^N = c_1\nabla_{K(ABC}\nabla^K{}_{DEF)}\mu^N,$$
$$\Box_{ABCD}\mu^N = c_2\nabla_{KJ(AB}\nabla^{KJ}{}_{CD)}\mu^N, \qquad \Box_{AB}\mu^N = c_3\nabla_{KJL(A}\nabla_{B)}{}^{KJL}\mu^N,$$
$$\Box\mu^N = c_4\nabla_{KJLM}\nabla^{KJLM}\mu^N,$$

where $c_1 = 2$, $c_2 = \frac{12}{7}$, $c_3 = \frac{4}{5}$ and $c_4 = \frac{1}{5}$.

Proof: Standard calculations. □

By the symmetry of $\nabla_{[a}\nabla_{b]}\mu^N$, the curvature-related operators are $\Box_{ABCDEF}$ and $\Box_{AB}$.

Definition 9.12 *The curvature tensors* $X_{ABCDEF}{}^{SK} = X_{(ABCDEF)}{}^{SK}$*, and* $X_{AB}{}^{SK} = X_{(AB)}{}^{SK}$ *are defined by*

$$\Box_{ABCDEF}\mu^S = X_{ABCDEF}{}^{SK}\mu_K,$$
$$\Box_{AB}\mu^S = X_{AB}{}^{SK}\mu_K.$$

Lemma 9.13 *The tensors* $X_{ABCDEFPQ}$ *and* X_{ABPQ} *are symmetric in* PQ.

Proof: Use the compatibility of ∇_{ABCD} with the metric ϵ_{AB}. □

Lemma 9.14 *One has* $X_{ABCDEF}{}^{GH} = 0$ *and* $X_{AB}{}^{CD} = s\epsilon_{(A}{}^{C}\epsilon_{B)}{}^{D}$ *for some constant* s.

Proof: The Riemannian curvature $R_{abc}{}^{d}$ can be written as

$$R_{a\ c}^{\ b\ d} = 8\epsilon_{(A}{}^{(E}X_{BCD)}{}^{FGH)}{}_{(L}{}^{(K}\epsilon_{M}{}^{R}\epsilon_{P}{}^{S}\epsilon_{Q)}{}^{T)} + \epsilon_{(A}{}^{(E}\epsilon_{B}{}^{F}\epsilon_{C}{}^{G}X_{D)}{}^{H)}{}_{(L}{}^{(K}\epsilon_{M}{}^{R}\epsilon_{P}{}^{S}\epsilon_{Q)}{}^{T)}.$$

By the symmetry $R_{abcd} = R_{cdab}$, one has $X_{AB}{}^{CD} = s\epsilon_{(A}{}^{C}\epsilon_{B)}{}^{D} + \Phi_{AB}{}^{CD}$, where $\Phi_{ABCD} = \Phi_{(ABCD)}$ and $X_{ABCDEF}{}^{GH} = 0$. Next, by the integrability condition in Appendix (Lemma 9.18), we have $\Phi_{ABCD} = 0$. □

9.3 The Transform

The part of $Œ(\mathcal{E}_{\eta}^{\bullet,0}(\eta^*\mathcal{O}(n)))$ complex on F that interests us is

$$\to Œ(n) \xrightarrow{\ \mathrm{A}\ } Œ_{(ABC)}(n+1) \xrightarrow[\mathrm{C}]{\ \mathrm{B}\ } \begin{matrix} Œ(n+2) \\ \oplus \\ Œ_{(BCEF)}(n+2) \end{matrix}$$

where

$$\mathrm{A} = -\pi^K\nabla_{ABCK} + n\rho'\sigma_{(A}\pi_B\pi_{C)}, \tag{9.2}$$
$$\mathrm{B} = -\pi^K\nabla^{ABC}{}_K + (n+1)\rho'\sigma^{(A}\pi^B\pi^{C)}, \tag{9.3}$$
$$\mathrm{C} = -\pi^K\nabla_K{}^A{}_{EF} + \frac{2}{3}\rho\epsilon_{(E}{}^A\pi_{F)} + \frac{n+1}{3}\rho'\sigma^A\pi_E\pi_F + \frac{2(n+1)}{3}\rho'\pi^A\sigma_{(E}\pi_{F)}, \tag{9.4}$$

where ρ, ρ' are two constants, σ_A is a section of $\text{Œ}_A(-1)$ and $\sigma_A \pi^A = 1$. See Appendix for details.

Note: While ρ' and σ_A depend on the scaling of π_A, the constant ρ is invariant under any rescaling of π_A. Without loss of generality, we shall assume $\rho = 1$, cf. Appendix.

The Isomorphisms

We next discuss the Penrose transform for $H^1(\mathbb{CP}_2\backslash\mathbb{RP}_2, \mathcal{O}(n))$ in three cases: $n \geq -1$, $n \leq -3$ and $n = -2$, using the spectral sequence (5.27). See Chapter 8 for the structures of spectral sequences which we omit here.

Proposition 9.15 *For $n \geq -1$, we have the following isomorphism*

$$H^1(\mathbb{CP}_2\backslash\mathbb{RP}_2, \mathcal{O}(n)) \cong \frac{\left\{\begin{array}{l} \nabla^{ABC(G}\psi_{ABC}{}^{P\ldots T)} + (n+1)\psi^{(PQ}{}_C{}^{|C|\ldots G)} = 0 \ \& \\ \nabla^{A(G}{}_{(EF}\psi_{|A|BC)}{}^{P\ldots T)} - \frac{2(n+1)}{3}\epsilon_{(B}{}^{(P}\psi^Q{}_{CEF)}{}^{R\ldots TG)} \\ +\frac{n+1}{3}\epsilon_{(B}{}^{(P}\epsilon_C{}^Q\psi_{EF)M}{}^{|M|\ldots G)} + \frac{2}{3}\epsilon_{(B}{}^{(P}\psi_{CEF)}{}^{Q\ldots TG)} = 0 \end{array}\right\}}{\left\{\nabla_{ABC}{}^{(P}\phi^{Q\ldots T)} + n\epsilon_{(A}{}^{(P}\epsilon_B{}^Q\phi_{C)}{}^{R\ldots T)}\right\}},$$

where $\psi_{ABC}{}^{P\ldots T} \in \Gamma(M^5, \mathcal{E}_{(ABC)} \otimes \odot^{n+1}\mathcal{E}^A)$, $\phi^{Q\ldots T} \in \Gamma(M^5, \odot^n\mathcal{E}^A)$ and $\odot^{-1}\mathcal{E}^A$ is taken to be vacuous.

Proof: The spectral sequence (5.27) yields

$$H^1(\mathbb{CP}_2\backslash\mathbb{RP}_2, \mathcal{O}(n)) \cong \frac{\mathrm{Ker}(\tau_*\mathrm{B}) \cap \mathrm{Ker}(\tau_*\mathrm{C})}{\mathrm{Im}(\tau_*\mathrm{A})}.$$

By the commutativity of the following diagram

$$\begin{tikzcd}
\text{Œ}(n) \arrow[r, "\mathrm{A}"] & \text{Œ}_{(ABC)}(n+1) \\
\mathcal{E}^{(\overbrace{Q\ldots T}^{n})} \arrow[u, "\pi_Q\ldots\pi_T"'] \arrow[r, "\nabla_{ABC}{}^P + \widetilde{\Phi}"] & \mathcal{E}_{(ABC)}{}^{(PQ\ldots T)} \arrow[u, "\pi_P\pi_Q\ldots\pi_T"']
\end{tikzcd}$$

$\widetilde{\Phi}$ here is computed to be

$$\widetilde{\Phi}\phi^{Q\ldots T} = n\epsilon_{(A}{}^{(P}\epsilon_B{}^{Q}\phi_{C)}{}^{R\ldots T)}.$$

Similarly the following commutative diagram

$$\begin{tikzcd}
\text{Œ}_{(ABC)}(n+1) \arrow[r, "\mathrm{B}"] & \text{Œ}(n+2) \\
\mathcal{E}_{(ABC)}{}^{(P\ldots T)} \arrow[u, "\pi_P\ldots\pi_T"'] \arrow[r, "\nabla^{ABCG}+\widetilde{\Phi}_1"] & \mathcal{E}^{(P\ldots TG)} \arrow[u, "\pi_P\ldots\pi_T\pi_G"']
\end{tikzcd}$$

gives

$$\widetilde{\Phi}_1\psi_{ABC}{}^{P\ldots T} = (n+1)\psi^{(PQ}{}_C{}^{|C|\ldots G)}.$$

Finally $\tau_*\mathrm{C}$ is determined by the commutativity of

$$\begin{tikzcd}
\text{Œ}_{(ABC)}(n+1) \arrow[r, "\mathrm{C}"] & \text{Œ}(n+2) \\
\mathcal{E}_{(ABC)}{}^{(P\ldots T)} \arrow[u, "\pi_P\ldots\pi_T"'] \arrow[r, "\nabla_{EF}{}^{AG}+\widetilde{\Phi}_2"] & \mathcal{E}_{(BCEF)}{}^{(P\ldots TG)} \arrow[u, "\pi_P\ldots\pi_T\pi_G"']
\end{tikzcd}.$$

Here $\widetilde{\Phi}_2$ is

$$\begin{aligned}\widetilde{\Phi}_2\psi_{ABC}{}^{P\ldots T} &= \tfrac{-2(n+1)}{3}\epsilon_{(B}{}^{(P}\psi^{Q}{}_{CEF)}{}^{R\ldots TG)} + \tfrac{n+1}{3}\epsilon_{(B}{}^{(P}\epsilon_C{}^{Q}\psi_{EF)M}{}^{|M|\ldots G)} \\ &\quad +\tfrac{2}{3}\epsilon_{(B}{}^{(P}\psi_{CEF)}{}^{Q\ldots TG)}.\end{aligned}$$

□

Proposition 9.16 *For $m \geq 1$, we have the following isomorphism*

$$H^1(\mathbb{CP}_2\backslash\mathbb{RP}_2, \mathcal{O}(-m-2)) \cong \left\{\nabla^{ABCE}\phi_{P\ldots E} + (m-1)\phi^{(AB}{}_{(P\ldots}\epsilon^{C)}{}_{R)} = 0\right\},$$

where $\phi_{P\ldots E} \in \Gamma(M^5, \odot^m\mathcal{E}_A)$.

Proof: The spectral sequence (5.27) gives

$$H^1(\mathbb{CP}_2\backslash\mathbb{RP}_2, \mathcal{O}(-m-2)) \cong \mathrm{Ker}(\tau_*^1\mathrm{A}),$$

where A is as mentioned before.

To work out $\tau_*^1\mathrm{A}$, we consider the following commutative diagram.

$$
\begin{tikzcd}
0 \arrow[r] & \text{Œ}(-m-2) \arrow[r, "\overbrace{\pi_P\ldots\pi_R\pi_E\pi_F}^{m+1}"] \arrow[d, "-\pi^K\nabla_{ABCK}-(m+2)\rho'\sigma_{(A}\pi_B\pi_{C)}"] & \text{Œ}_{(P\ldots F)}(-1) \arrow[r, "\pi^F"] \arrow[d, "\nabla_{ABC}{}^E+\hat{\Phi}"] & \mathcal{E}_{(P\ldots E)} \arrow[r] \arrow[d, "\nabla_{ABC}{}^E+\widetilde{\Phi}"] & 0 \\
0 \arrow[r] & \text{Œ}_{(ABC)}(-m-2) \arrow[r, "\pi_P\ldots\pi_R\pi_F"] & \text{Œ}_{(ABC)(P\ldots RF)}(-1) \arrow[r, "\pi^F"] & \mathcal{E}_{(ABC)(P\ldots R)} \arrow[r] & 0.
\end{tikzcd}
$$

For the first square to commute, we can have

$$\hat{\Phi}\phi_{P\ldots F} = P_{ABCP\ldots RF},$$

where

$$
\begin{aligned}
P^{ABC}{}_{P\ldots RF} = {} & m\phi^{(AB}{}_{(P\ldots}\epsilon^{C)}{}_{F)} + 3\sigma^{(A}\pi^B\epsilon^{C)E}\phi_{P\ldots EF} + \pi^A\pi^B\pi^C\zeta^E\phi_{P\ldots EF} \\
& +(\rho'-1)(\sigma^{(A}\pi^B\pi^{C)}\pi^E + \sigma^{(A}\pi^B\pi^{C)}\sigma^E)\phi_{P\ldots REF}.
\end{aligned}
$$

See Appendix for definition of ζ_A.

Next, for the second square to commute, we obtain

$$\widetilde{\Phi}\phi_{P\ldots E} = Q_{ABCP\ldots R},$$

where

$$Q^{ABC}{}_{P\ldots R} = (m-1)\phi^{(AB}{}_{(P\ldots}\epsilon^{C)}{}_{R)}.$$

□

Proposition 9.17 *One has the following isomorphism*

$$H^1(\mathbb{CP}_2\backslash\mathbb{RP}_2, \mathcal{O}(-2)) \cong \left\{\Box\phi = k\phi \quad and\ \Box_{ABCD}\phi + \frac{8}{7}\nabla_{ABCD}\phi = 0\right\},$$

where $\phi \in \Gamma(M^5, \mathcal{E})$ and k is a constant which we have not computed.

Proof: The spectral sequence (5.27) follows the same pattern as that for the $\mathcal{O}\{-1\}$ case of $\mathbb{R}^5$, cf. Chapter 8. The only difference is on the differential maps.

Consider the following commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \text{Œ}(-2) \arrow[r, "\pi_K"] \arrow[d, "\wr\|"] & \text{Œ}_K(-1) \arrow[r, "\pi^K"] \arrow[d, "\nabla^K{}_{ABC}+\hat{\Phi}"] & \mathcal{E} \arrow[r] \arrow[d, "D"] & 0 \\
\cdots \arrow[r] & \text{Œ}(-2) \arrow[r, "\mathrm{A}"] & \text{Œ}_{(ABC)}(-1) \arrow[r, "\mathrm{B}", "\mathrm{C}"'] & \begin{array}{c}\mathcal{E}\\ \oplus \\ \mathcal{E}_{(BCEF)}\end{array} \arrow[r] & \cdots
\end{tikzcd}
$$

where A, B, C are as defined in (9.2)—(9.4), with n being -2.

From the commutativity of the first square, we can obtain

$$\hat{\Phi}\phi_K = 3\sigma_{(A}\pi_B\epsilon_{C)}{}^K\phi_K + (\rho' - 1)(\sigma_{(A}\pi_B\sigma_{C)}\pi^K + \sigma_{(A}\pi_B\pi_{C)}\sigma^K)\phi_K + \pi_A\pi_B\pi_C\zeta^K\phi_K.$$

Write $D = D_1 + D_2$, where D_1 and D_2 are the maps into $\mathcal{E}$ and $\mathcal{E}_{(BCEF)}$ respectively.

For D_1 we have

$$\begin{aligned} D_1(\pi^K\phi_K) &= \mathrm{B}(\nabla^K{}_{ABC} + \hat{\Phi})\phi_K \\ &= \tfrac{1}{2}\nabla_{ABCD}\nabla^{ABCD}(\pi^K\phi_K) + (\tfrac{-15}{8}s - \tfrac{1}{3})(\pi^K\phi_K) \\ &\quad + \tfrac{1}{2}\phi_K(\pi^A\pi^B\pi^C\pi^D\nabla_{ABCD}\zeta^K). \end{aligned}$$

As the last term of the RHS involves no differential of ϕ_K, $\pi^A\pi^B\pi^C\pi^D\nabla_{ABCD}\zeta^K$ is necessarily proportional to π^K, and we obtain that

$$D_1\phi = \frac{5}{2}(\Box\phi - k\phi).$$

For D_2, we have

$$\begin{aligned} D_2(\pi^K\phi_K) &= \mathrm{C}(\nabla^K{}_{ABC} + \hat{\Phi})\phi_K \\ &= \tfrac{7}{24}\Box_{BCEF}(\pi^K\phi_K) + \tfrac{1}{3}\nabla_{BCEF}(\pi^K\phi_K) + \Pi_{BCEF}, \end{aligned}$$

where Π_{BCEF} is some spinor field involving no differential of ϕ_K. The only possible Π_{BCEF} is zero. We thus have

$$D_2\phi = \frac{7}{24}(\Box_{ABCD}\phi + \frac{8}{7}\nabla_{ABCD}\phi).$$

Using arguments similar to that in the proof of Proposition 2.12, we then obtain the result. □

Appendix: Some formulae

On $\mathbb{M}^5$, let $\mathcal{O}^A$ be the bundle given by $\mathcal{O}^A \mid_x = H^0(C_x, \mathcal{O}(1))$, where $C_x \cong \mathbb{P}_1$ is the conic in $\mathbb{P}_2$ corresponding to $x \in \mathbb{M}^5$, and we use ∇_{ABCD} to denote the spinorial connection corresponding to the Levi-Civita connection.

Lemma 9.18 *The integrability condition for a spinor field π_A to integrate to give an α-surface is*

$$\pi^A \pi^D \pi_N \nabla_{ABCD} \pi^N = 0. \tag{9.5}$$

Proof: The integrability condition is equivalent to that the Lie bracket of two vector fields both of the form $\pi^{(A}\omega^{BCD)}$ will be again of the same form. As ∇_{ABCD} is the Levi-Civita connection, we have

$$\omega^{ABC}\pi^D \nabla_{ABCD}(\kappa^{(EFM}\pi^{N)}) - \kappa^{ABC}\pi^D \nabla_{ABCD}(\omega^{(EFM}\pi^{N)}) = \phi^{(EFM}\pi^{N)},$$

where ω^{ABC}, κ^{ABC}, and ϕ^{ABC} are some sections of $\mathcal{O}^{(ABC)}$. The result follows after some calculations. □

Therefore on $\mathbb{F}$, there is a spinor field π_A such that (9.5) is satisfied. By restriction of $\mathbb{M}^5$ to M^5, there is a spinor field π_A on F also, with the integrability condition satisfied.

Definition 9.19 *Define $\rho \in \Gamma(M^5, \mathcal{E})$ by*

$$\pi_N \pi^D \nabla_{ABCD} \pi^N = \rho \pi_A \pi_B \pi_C, \tag{9.6}$$

which is equivalent to (9.5), cf. [29].

Note that ρ is invariant under any rescaling of π_A and is a constant as M^5 is a symmetric space. However, if we rescale the metric ϵ_{AB} by $\epsilon_{AB} \longmapsto c\epsilon_{AB}$ where $c \neq 0$ is a constant, then $\rho \longmapsto c^{-2}\rho$. Therefore, without loss of generality, we can assume that $\rho = 1$.

Lemma 9.20 *One has*

$$\pi^D \nabla_{ABCD} \pi^N = \rho \pi_{(A} \pi_B \epsilon_{C)}{}^N + \rho' \sigma_{(A} \pi_B \pi_{C)} \pi^N, \tag{9.7}$$

where ρ' is some constant, σ_A is a section of $\mathbb{E}_A(-1)$ and $\sigma_A \pi^A = 1$.

Proof: The integrability condition for

$$\pi^A \pi^B \pi^C \nabla_{ABCD} \pi^N = 0 \tag{9.8}$$

is

$$\pi^A \pi^B \pi^C \pi^D \pi^E \pi^F X_{ABCDEF}{}^{NK} \pi_K = 0,$$

which is trivial as we have $X_{ABCDEF}{}^{NK} = 0$. We can thus choose a scaling of π_A such that (9.8) holds. The condition (9.8) is equivalent to

$$\pi^C \pi^D \nabla_{ABCD} \pi^N = \lambda \pi_A \pi_B \pi^N, \tag{9.9}$$

where λ is a section of $\mathcal{E}$. Under a rescaling $\pi_A \longmapsto \Omega \pi_A$ such that $\pi^C \pi^D \nabla_{ABCD} \Omega = \delta \pi_A \pi_B \Omega$, one has

$$\lambda \longmapsto \lambda + \delta.$$

We shall consider only scalings of π_A such that λ is a constant.

From (9.6) and (9.9) one has

$$\pi^D \nabla_{ABCD} \pi^N = \rho \pi_{(A} \pi_B \epsilon_{C)}{}^N + \kappa_{(A} \pi_B \pi_{C)} \pi^N,$$

where κ_A is a section of $\mathbb{E}_A(-1)$ and $\lambda = \frac{1}{3}(\rho + \kappa_A \pi^A)$. Take a generic scaling such that $\kappa_A \pi^A \neq 0$. Then one can write $\kappa_C = \rho' \sigma_C$ such that ρ' and σ_A are as required. □

Lemma 9.21 *The spinor field $\nabla_{ABCD} \pi^N$ is of the following form*

$$\nabla_{ABCD} \pi^N = 4\rho \pi_{(A} \pi_B \pi_C \epsilon_{D)}{}^N + (2\rho' - 2\rho) \sigma_{(A} \pi_B \sigma_C \pi_{D)} \pi^N + \pi_A \pi_B \pi_C \pi_D \zeta^N, \tag{9.10}$$

where ζ^A is a section of $\mathbb{E}^A(-3)$.

Proof: It is equivalent to (9.7). □

Lemma 9.22 *One has*

$$\pi^D\nabla_{ABCD}\sigma^N = -(\rho+\rho')\sigma_{(A}\pi_B\sigma_{C)}\sigma^N - 3\rho\sigma_{(A}\sigma_B\pi_{C)}\pi^N + \gamma\pi_A\pi_B\pi_C\pi^N, \quad (9.11)$$

where γ is a section of $\mathcal{E}(-4)$.

Proof: From $\pi^D\nabla_{ABCD}(\sigma_N\pi^N)=0$, we can obtain

$$\pi^D\nabla_{ABCD}\sigma_N = -(\rho+\rho')\sigma_{(A}\pi_B\sigma_{C)}\sigma_N + Z_{ABC}\pi_N,$$

where Z_{ABC} is a section of $\mathcal{E}_{(ABC)}(-1)$.

Now consider the following part of $\mathcal{E}(\mathcal{E}^{\bullet,0}_\eta(\eta^*\mathcal{O}(n)))$ complex.

$$\to \mathcal{E}(n) \xrightarrow{\ \mathrm{A}\ } \mathcal{E}_{(ABC)}(n+1) \xrightarrow[\mathrm{C}]{\mathrm{B}} \begin{matrix}\mathcal{E}(n+2)\\ \oplus \\ \mathcal{E}_{(BCEF)}(n+2).\end{matrix}$$

For A to be well-defined, acting on $\mathcal{E}(n)$, we have

$$\mathrm{A} = -\pi^K\nabla_{ABCK} + n\rho'\sigma_{(A}\pi_B\pi_{C)}.$$

For B and C, the parts involving pure differentiation are $-\pi^K\nabla^{ABC}{}_K$ and $-\pi^K\nabla_K{}^A{}_{EF}$ respectively. By the requirement that B∘A and C∘A are zero, B and C are uniquely determined:

$$\begin{aligned}\mathrm{B} &= -\pi^K\nabla^{ABC}{}_K + (n+1)\rho'\sigma^{(A}\pi^B\pi^{C)}\\ \mathrm{C} &= -\pi^K\nabla_K{}^A{}_{EF} + \tfrac{2}{3}\rho\epsilon_{(E}{}^A\pi_{F)} + \tfrac{n+1}{3}\rho'\sigma^A\pi_E\pi_F + \tfrac{2(n+1)}{3}\rho'\pi^A\sigma_{(E}\pi_{F)},\end{aligned}$$

where we have chosen scalings of π_A such that ρ' is a constant.

From the vanishing of B∘A and C∘A we also obtain

$$\begin{cases}\pi^A\pi^B\pi^C\nabla_{ABC}{}^K\sigma_K = 0\\ \pi^D\pi_{(B}\pi_C\nabla_{EF)DK}\sigma^K - 2\pi^D\pi_A\pi_{(B}\nabla_{|D|}{}^A{}_{CE}\sigma_{F)} + 6\rho\sigma_{(B}\pi_C\pi_E\pi_{F)} = 0.\end{cases}$$

After some calculations, we obtain the result. □

Lemma 9.23 *The spinor field $\nabla_{ABCD}\sigma^N$ is of the form*

$$\begin{aligned}\nabla_{ABCD}\sigma^N &= -4\rho\sigma_{(A}\pi_B\sigma_C\epsilon_{D)}{}^N - (6\rho+2\rho')\sigma_{(A}\sigma_B\sigma_C\pi_{D)}\sigma^N + 4\gamma\sigma_{(A}\pi_B\pi_C\pi_{D)}\pi^N\\ &\quad +\pi_A\pi_B\pi_C\pi_D\eta^N, \qquad (9.12)\end{aligned}$$

where η^A is a section of $\mathcal{E}^A(-5)$ and $\sigma_K\zeta^K + \eta_K\pi^K = 0$.

Proof: This is equivalent to (9.11). $\square$

Chapter 10

A Generalized Correspondence

In this chapter we consider the correspondence between complex surfaces which admit curves of normal bundle $\mathcal{O}(4)$ and their moduli spaces, of which $\mathbb{C}^5$ and $\mathbb{M}^5$ are two examples. In Section 10.2, we generalize this to a correspondence between a complex surface admitting curves with normal bundles $\mathcal{O}(m)$, ≥ 1 and its moduli space. This is motivated by the generalization of self-dual manifolds to torsion-free QCF manifolds, cf. [2]. It is hoped that the moduli spaces of this chapter may be of interest as reduction by $(2n-1)$-dimensional symmetry groups of $4n$-dimensional QCF manifolds (cf. the reduction of self-dual manifolds to EW spaces [19]). One can also apply the holomorphic Penrose transform to these correspondences, but we have not computed details.

10.1 Complex surfaces with curves of normal bundle $\mathcal{O}(4)$ and their moduli spaces

Definition 10.1 *Let X be a complex 5-manifold with a projective structure $[D]$ and an isomorphism between TX and $\odot^4\mathcal{O}^A$, where $\mathcal{O}^A$ is a holomorphic rank 2 vector bundle over X, then*

(1) a vector V at $x \in X$ is said to be 4-null if it is of the form $\pi^A\pi^B\pi^C\pi^D$,

(2) a hypersurface is said to be 4-null if its normal vectors everywhere are 4-null,

(3) an 'α-surface' is a totally geodesic (with respect to [D]) 4-null hypersurface.

Proposition 10.2 *There is a one-one correspondence between:*

(a) *Complex surfaces T with rational curves with normal bundle $\mathcal{O}(4)$,*

(b) *Complex 5-manifolds X with*

(i) *a projective structure $[D]$,*

(ii) *an isomorphism $TX \cong \odot^4\mathcal{O}^A$,*

(iii) *the existence of a two dimensional family of α-surfaces.*

Proof: The proof will be in four steps:

1) Given a T, one can construct an X with properties (i),(ii),(iii): Given T with an embedded curve Y with normal bundle $\mathcal{O}(4)$, by Kodaira's theorem (see Theorem 0.16) , Y belongs to a locally complete family of curves $\{Y_x : x \in X\}$, where the parameter space X is a complex manifold and there is a canonical isomorphism $T_xX \cong H^0(Y_x, \mathcal{O}(4)) \cong \mathbb{C}^5$.

(i) Generalizing Hitchin's argument in [16], we define a projective structure on X as follows: Given four points in T, blow up T at these four points (with multiplicity if degenerate) obtaining a surface $\widetilde{T}$ which has embedded curves with trivial normal bundle. By Kodaira's theorem, we have a one parameter family of such embedded curves and when projected down to T, they all pass through those four points. We then say a curve γ in X is a *distinguished curve* if points on γ correspond to rational curves in T passing through the same four points. One thus has a distinguished family of curves. Furthermore they are indeed geodesics of a genuine projective structure, see [16] for a sheaf cohomological argument.

(ii) Define $\mathcal{O}^A$ on X by $\mathcal{O}^A_x := H^0(Y_x, \mathcal{O}(1))$, then by $T_xX \cong H^0(Y_x, \mathcal{O}(4))$, we have an isomorphism $TX \cong \odot^4\mathcal{O}^A$.

(iii) Consider a hypersurface in X corresponding to all curves through a point $p \in T$. This surface is totally geodesic with respect to $[D]$ as geodesics of $[D]$ stay in such a hypersurface by definition. To see that it is 4-null, we write a tangent vector Ψ^{ABCD} to it at some point as $\Psi^{ABCD}\pi_A\pi_B\pi_C\pi_D = (z-p)(az^3+bz^2+cz+d)$, meaning a section of $H^0(Y_x, \mathcal{O}(4))$ which vanishes at p. Supposing p has coordinate $[\tau_0, \tau_1]$ and plugging it into the equation, then one has $\Psi^{ABCD}\tau_A\tau_B\tau_C\tau_D = 0$. That is, the hypersurface has normal $\tau^A\tau^B\tau^C\tau^D$. Therefore such a hypersurfaces is an α surface. It is not difficult to check that every α-surface of X arises this way.

2) Given an X, one can construct a T with embedded curves having normal bundle $\mathcal{O}(4)$: Given X, let T be the space of α-surfaces in X. For every $x \in X$, there are $\mathbb{CP}_1$ worth of α-surfaces through x, with normals at x being $\pi^A\pi^B\pi^C\pi^D$. Thus x corresponds to a curve in T. Let x' be a point near x corresponding to another curve in T, then x, x' are connected by a geodesic, with respect to $[D]$, which assigns a tangent direction $[v]$ at x. The orthogonal space $v^{\perp} = \{u \in T_xX, \langle u, v\rangle_{[g]} = 0\}$ is a four dimensional vector subspace in T_xX. It will intersect the 4-null zone N, the set of 4-null vectors, at four directions. Then by the existence of α-surfaces, we obtain four α-surfaces. Thus the curve associated to x has normal bundle $\mathcal{O}(4)$.

(3) Applying 1) then 2), one gets the original T back: A point $p \in T$ corresponds to all curves in T through p. By (1) this is an α-surface in X. Then by (2), this α-surface corresponds to a point in T', a new complex surface. The new T' is isomorphic to T as they are both the space of α-surfaces of $\mathbb{M}$.

(4) Applying 2) then 1), one obtains the original X with the same structure: A point x in X corresponds to all α-surfaces through x, namely a curve in T, which in turn is parameterized by a point in X', a new space. That is, the new space X' is biholomorphic to the original X. To show that X' has the same structure as X, we need:

i) The new projective structure $[D']$ agrees with $[D]$: Let γ be a geodesic with respect to $[D]$ and x, x' are points on γ. Then there are four α-surfaces through them, namely the intersection of Y_x and $Y_{x'}$, as is discussed in the proof of (2). By the fact that α-surfaces are totally geodesic, all points in γ lie in these four α-surfaces. That is, all the corresponding curves in T pass through the same four points in T. Thus γ is also a geodesic with respect to $[D']$ by definition.

ii) The isomorphism $TX \cong \odot^4\mathcal{O}^A$ is the same for both X and X': Given $\mathcal{O}^A$ on X, a spinor π^A at $x \in X$ can be interpreted as a section of $H^0(Y_x, \mathcal{O}(1))$, where $\pi^A\pi^B\pi^C\pi^D$ gives rise to the α-surface corresponding to that root on Y_x. We thus have an isomorphism $\mathcal{O}_x^A \cong H^0(Y_x, \mathcal{O}(1))$. Since the $\mathcal{O}_x^A$ on X' is by definition $H^0(Y_x, \mathcal{O}(1))$ we have that the $\mathcal{O}^A$ for X and the $\mathcal{O}^A$ for X' are isomorphic.

iii) They have the same α-surfaces: This is certainly true as an α-surface α_p in X and an α-surface α'_p in X' both correspond to all curves passing though $p \in T$.

We thus obtain the one-one correspondence between (a) and (b). □

Proposition 10.3 *There exists a unique conformal structure $[g]$ on X such that at any point $x \in X$, 3-null vectors (vectors of the form $\pi^{(A}\pi^B\pi^C\omega^{D)}$) are null.*

Proof: Prescribing a null cone in T_xX, $x \in X$, is the same as having a quadratic equation in the coefficients of $az^4 + bz^3 + cz^2 + dz + e \in H^0(\mathbb{P}_1, \mathcal{O}(4))$. Now a vector $\pi^{(A}\pi^B\pi^C\omega^{D)}$ corresponds to a section $\gamma(z-\alpha)^3(z-\beta)$. It is a straightforward calculation to show that there exists one and only one quadratic equation in a, b, c, d, e, namely $c^2 - 3bd + 12ae = 0$, such that coefficients of $\gamma(z-\alpha)^3(z-\beta)$ is a solution to that equation for all α, β, and γ. One thus has a uniquely prescribed conformal structure. □

10.2 A Generalization

Proposition 10.4 *For $m \geq 1$, there is a 1-1 correspondence between:*

(a) *Complex surfaces T with rational curves with normal bundle $\mathcal{O}(m)$,*

(b) *Complex (m+1)-manifolds X with*

- **(i)** *a projective structure $[D]$,*
- **(ii)** *an isomorphism $TX \cong \odot^m \mathcal{O}^A$,*
- **(iii)** *the existence of a two dimensional family of α-surfaces,*

where α-surfaces are defined to be totally geodesic m-null hypersurfaces, where 'm-nullness' is defined in the same way as that in Definition 10.1.

The proof is essentially the same as that for the previous proposition. When $m = 4$, Proposition 10.4 reduces to Proposition 10.2. When $m = 2$, it reduces to Hitchin's correspondence.

Definition 10.5 *When $m = 2n$, one can give X a conformal structure: a vector at $x \in X$ is null if its corresponding section $a_0 z^{2n} + a_1 z^{2n-1} + \ldots + a_{2n-1} x + a_{2n}$ of $\mathcal{O}(2n)$ satisfies*

$$\frac{a_n^2}{\binom{2n}{n}} + 2\sum_{i=1}^{n}(-1)^i \frac{a_{n+i}a_{n-i}}{\binom{2n}{n-i}} = 0, \tag{10.1}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$.

Proposition 10.6 *With respect to this conformal structure, vectors of the form* $\pi^{(\overbrace{A \ldots \pi^{C}}^{n+1}\overbrace{\tau^{D} \ldots \omega^{E)}}^{n-1}}$ *are null.*

Proof: Let $f(z) = a_0 z^{2n} + \ldots + a_{2n}$. If it has an $(n+1)$-ple root z_0, then

$$f(z_0) = f^{(1)}(z_0) = \ldots = f^{(n)}(z_0) = 0,$$

where $f^{(k)}(z_0) := \frac{\partial^k f}{\partial z^k}(z_0)$. In particular

$$f^{(n)}(z_0)^2 + 2\sum_{i=1}^{n}(-1)^i f^{(n+i)}(z_0) f^{(n-i)}(z_0) = 0. \tag{10.2}$$

It is a straightforward calculation to check that, in (10.2), coefficients of z_0^k's, $k \neq 0$, are all zero, and one is left with (10.1). $\square$

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